A HANDBOOK

ON THE

TEETH OF GEARS,

THEIR CURVES, PROPERTIES,

AND

PRACTICAL CONSTRUCTION.

By GEORGE B. GRANT, M.E.

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THE TEETH OF
GEAR WHEELS.

INTRODUCTION.

Few mechanical subjects have attracted the attention of scientific men to such an extent, or are so intimately connected with mathematics, as the proper construction of the teeth of gear wheels, and, as a consequence, few can show such an advance as has here been made, from the rough cog wheel of not many years ago, to the neat cut gear of the present day.

It is not apparent wherein much further improvement is needed in our knowledge of the theory of the subject, but it is evident that much remains to be done towards its practical application, and to induce the working mechanic to understand and use the improvements that have been developed by the mathematician and the inventor. The theory seems to be full and well nigh perfect, but the mill-wright and the machinist still clings to imperfect rules and clumsy devices that should have been forgotten years ago, and few workmen have a clear knowledge of even the rudiments of the science which it is their business to apply to practical purposes.

It is the mathematical and scientific character of the subject that makes it so difficult to the practical man, who can understand but little of it as it is commonly presented in elaborate treatises or encyclopædias, and who takes but little interest in the study of a matter that bristles with strange characters and technical terms.

I have here undertaken to address the workman as well as the man of science, and have felt obliged to leave out nearly everything that cannot be treated in a plain, descriptive manner, to use language that any intelligent man can understand, and to refer to more pretentious works than this for demonstrations, or unessential details.

A volume of a thousand pages would not properly present the whole subject, and this little pamphlet can deal only with the main principles and prominent points. It is not a treatise, it is a hand-book that does not pretend to cover the whole ground, and its principal object is to present the new odontographs, which I believe to be superior to those heretofore in use for the purpose of designing the teeth of gear wheels.

FIRST PRINCIPLES.

The original gear wheel had pins or projections for teeth, of any form that would serve the general purpose and communicate an unsteady motion from one wheel to another.

FIG. 1.
THE ORIGINAL GEAR WHEEL.

The perfect gear wheel is the friction wheel, communicating a smooth, uniform, rolling motion, by means of the frictional contact of its surface. It is, in fact, a gear wheel with a great many very small, weak, and irregular teeth.

FIG. 2. FRICTION WHEELS.

The whole aim and object of the science of the teeth of gear wheels is to increase the size and strength of these teeth without destroying the uniformity of the motion they transmit, and this is accomplished by studying the shape of the teeth, and giving their bearing surfaces the curved outline that is found to produce the desired result.

There are an infinite number of curves that will meet the requirement, but only two, the epicycloid and the involute, are of any practical importance, or in actual use.
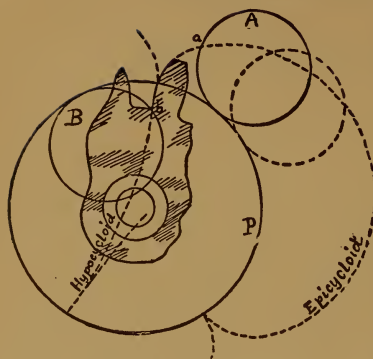
THE EPICYCLOIDAL TOOTH.

FIG. 3. THE EPICYCLOIDAL TOOTH.

The epicycloidal or double curve tooth has its bearing surface formed of two curves, meeting at the pitch line P, which corresponds to the working circle of the perfect gear wheel of fig. 2.

If a small circle, a, be rolled around on the outside of the pitch circle, p, a fixed tracing point, a, in its edge, will trace out the dotted line called an epicycloid, and a small part of this curve near the pitch line, usually one sixth of its full height, forms the face of the tooth.

Similarly, if a small circle, B, be rolled around on the inside of the pitch line, its tracing point, b, will describe the internal epicycloid, or hypocycloid, a small portion of which is used for the flank of the tooth.

If a projection be formed on the friction wheel fig. 4, the curved outline of which is a whole epicycloid E, and a depression be formed in the wheel N having a whole hypocycloid H for its outline, then, if both curves have been formed by the same describing circle B, it can be mathematically demonstrated that the two curves will just touch and slide on each other, without separating or intersecting, while the two friction wheels roll together.

The reverse of this fact is also true, that, if one wheel drives another by means of an epicycloidal projection on it working against a hypocycloidal depression in the other, both curves being formed by the same describing circle, the two wheels will roll together as uniformly as if driven by
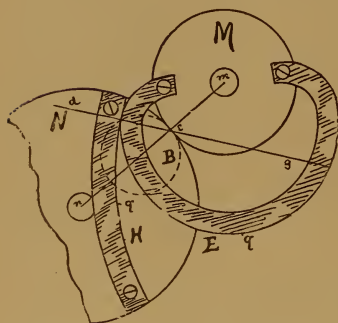

FIG. 4 WHOLE EPICYCLOIDAL TEETH.

frictional contact, and it is this peculiar property of the epicycloid that gives it its value for the purpose in hand.

The pressure acting between the two curves is in the direction of the line dg, is direct only at the start, and becomes more and more oblique, until, when the middle points, q q, come together, and beyond, there is no driving action at all. This defect forbids the use of the whole curve and we can use but a small portion of it near the pitch line. Another projection and depression must be formed so near the first that they will come into working position before the first pair are out of contact, thus forming the theoretically perfect but incomplete gears of fig. 5.
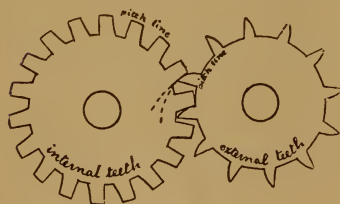

FIG. 5. INCOMPLETE EPICYCLOIDAL TEETH.

Practical requirements still further modify the apparent shape of the tooth, for it is desirable that the wheels shall work in either direction, and that they shall be interchangeable, so that any one of a set of several shall work with any other of that set.

This can be accomplished only by making the curves face both ways, and by putting both projections and depressions on each gear, thus forming the familiar tooth of fig. 3.

THE INTERCHANGEABLE SET.

If all the curves of a set of several gears, both the faces and the flanks of each gear, are described by the same rolling circle, the set will be interchangeable, and any one will work perfectly with any other.

This is a property of the greatest practical importance, and interchangeable sets should come into as universal use on heavy mill work as with cut gearing. It is the only system that will allow the use of a set of ready made cutters, and is therefore essential to the economical manufacture of cut gear wheels.

The diameter of the rolling circle is usually made half the diameter of the smallest gear of the set, and that gear will have straight radial lines for flanks.

The set in almost universal use and adopted for all the odontographs, has twelve teeth in its smallest gear, but there is a tendency to change this well established system, and create confusion for which the writer can see no adequate excuse, by the adoption of a pinion of fifteen teeth as the base or smallest gear. It may be admitted that as large a base as possible should be used, but the change from twelve to fifteen seems to be unwarranted in view of the confusion it creates by the abrupt change from an old and good rule to a new one that is a mere shade better, and the trouble it makes with small pinions of eight to twelve teeth.

RADIAL FLANK TEETH.

If the internal curves, or flanks, of a pair of gears that are to run together are on each radial straight lines described by a rolling circle of half its pitch diameter, and the rolling circle that describes the flanks of one gear is used to describe the faces of the other gear, then, the two gears will form a pair fitted to each other and not interchangeable with other gears.

This style of gear is very often used under the erroneous impression that it is the best possible form, and will give the least possible friction and thrust on the bearings, but the saving in friction over the interchangeable form would be an exceedingly difficult thing to measure by any practicable method, although it can be mathematically demonstrated to be a fact, and the slender roots of such teeth make them weaker and much inferior to the others. The odontograph figures show both a pair of these gears, and the same pair on the interchangeable plan, also, by the dotted lines on the former figure, the shapes as they would be on the interchangeable plan. It is plainly seen that the interchangeable faces are but a shade more rounding, while their flanks are so curved that the teeth are much stronger at the roots. The larger the describing circle, the less the theoretical thrust and friction, and if the flanks were formed by a describing circle of more than half the diameter of the gear, the teeth would be undercurved, the friction less, and their strength less, than that of the radial flank tooth.

In practical matters it is a good plan to give first place to practical points, and not to take too much notice of minute theoretical advantages, and there is no good reason, that will bear the test of experiment, for adopting the radial flank, non-interchangeable, and weak tooth, in preference to the strong tooth of the interchangeable system.

THE PITCH.

The pitch is a term used to designate the size of the tooth, and is either circular or diametral.

THE CIRCULAR PITCH or more properly the circumferential pitch, is the actual distance from tooth to tooth measured along the curve of the pitch line, and is expressed in inches, as ⅜ inch pitch, 1½ inch pitch, etc.

The table gives the proper pitch diameter of a gear of any given number of teeth, and one inch circular pitch. The tabular numbers must be multiplied by any other pitch that is in use.

Formerly, the circular pitch was the only one known, but it has deservedly gone out of use on cut gears, and it is hoped may soon be abandoned altogether. It is a clumsy, awkward, and troublesome device on either large or small work, having its origin in the ignorance of the past, and owing its

existence not to any perceptible merit, but to habit, and the natural persistence of an established custom.

With the circular pitch the relation between the pitch diameter of the gear, and the number of teeth on it, is fractional. If the diameter is a convenient quantity, such as a whole number of inches, the pitch must be an inconvenient fraction, and if the pitch is a handy part of an inch, the diameter will contain an unhandy decimal.

With the circular pitch there is no one length of tooth that is better than any other, and consequently there is no agreement upon that point. Each maker is at liberty to chose his own distance at random, and whatever he choses is as good as any other.

Its worst feature is that it leads to endless errors, for the average mechanic appreciates convenience more than accuracy, and will stretch his figures to suit his facts, with a botch as the common result.

A millwright figures out a diameter of 22.29 inches for a gear of one inch pitch and 70 teeth, and failing to make such a clumsy figure fit his work or his foot rule, and thinking a quarter of an inch or so to be of no importance, he lets it go at 22 whole inches. The same process on its mate of 15 teeth gives a 5 inch gear instead of one of 4.78 inches diameter, and the pair will never run or wear together properly. His only alternatives are to adopt the clumsy true diameters, or else use the clumsy figure .988 inch for his pitch.

Again, he is apt to apply a carpenter's rule directly to the teeth of the gear he is to repair or match, and naturally takes the nearest convenient fraction of an inch as his measurement, when the real pitch may be just enough different to spoil the job.

There is no reason whatever for using the circular pitch, unless the work to be done is to match work already in use.

THE DIAMETRAL PITCH is an immense improvement on the old fashioned circular pitch. It is not a measurement, but a number, or ratio. It is the number of teeth on the gear, for each inch of its pitch diameter, and its merit is that it establishes a convenient and manageable relation between these two principal elements, so that the calculations are of the simplest description and the results convenient and accurate.

The product of the pitch and the pitch diameter is equal to the number of teeth, and the number of teeth divided by the pitch is equal to the pitch diameter. A gear of 15 inches diameter and 2 pitch has 30 teeth, and a gear of 27 teeth of 4 pitch has a pitch diameter of $6\frac{3}{4}$ inches.

The rule that the length of the tooth is two pitch parts of an inch, $\frac{2}{4}$ or $\frac{1}{2}$ an inch for 4 pitch, $\frac{2}{2}$ or 1 inch for 2 pitch, etc. is so simple and so much better than any other that it is never disputed, and is in universal use.

The circular and diametral pitches are connected by the relation

$$c \times p = 3.1416.$$

or, the product of the circular and the diametral pitch is the number 3.1416.

THE ADDENDUM.

For reasons expressed above we can use but a small part of the epicycloidal curve near the pitch line, limiting it by a circle drawn at a distance inside or outside of the pitch line called the addendum. The outside limit need not be the same as the inside limit, but it is customary to make them equal.

When the diametral pitch is used, the length of the addendum is always one pitch part of an inch, as $\frac{1}{4}$th inch for 4 pitch, $\frac{1}{3}$rd inch for 3 pitch, etc. If we use the same proportion for circular pitches the addendum will be $\frac{1}{3}\cdot\frac{1}{1.1416}$ circular pitch, and the value $\frac{1}{3}$rd of the circular pitch may be adopted as the most convenient for use.

THE CLEARANCE.

Theoretically, the depression formed inside the pitch line should be only as deep as the projection outside of it is high, but to allow for practical defects in the making or in the adjustment of the teeth, and to provide a place for

dirt to lodge, the depression is always deeper than theory requires by an amount called the clearance. The amount of the clearance is arbitrary, but the sixteenth part of the depth of the tooth is a convenient and customary measure, or $\frac{1}{24}$th of the circular pitch, and 1 divided by 8 times the diametral pitch. The following tables will be convenient and save calculation:

CLEARANCE FOR CIRCULAR PITCHES.

Circular pitch.	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	1	$1\frac{1}{8}$	$1\frac{1}{4}$	$1\frac{3}{8}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2	$2\frac{1}{4}$	$2\frac{1}{2}$	3
Clearance.	.02	.03	.03	.04	.04	.05	.05	.06	.06	.07	.08	.09	.10	.12

CLEARANCE FOR DIAMETRAL PITCHES.

Diametral pitch.	6	5	4	$3\frac{1}{2}$	$3\frac{1}{4}$	3	$2\frac{3}{4}$	$2\frac{1}{2}$	$2\frac{1}{4}$	2	$1\frac{3}{4}$	$1\frac{1}{2}$	$1\frac{1}{4}$	1
Clearance.	.02	.03	.03	.04	.04	.04	.05	.05	.06	.06	.08	.09	.10	.12

THE BACKLASH.

When wooden cogs or rough cast teeth are used, the inevitable irregularities require that the teeth should not pretend to fit closely, but that the spaces should be larger than the teeth by an amount called the backlash. The amount of the backlash is arbitrary, but it is customary to make it about equal to the clearance.

Cut gears should have no allowance for backlash, and involute teeth need less backlash than epicycloidal teeth.

PITCH DIAMETERS.

FOR ONE INCH CIRCULAR PITCH.

For Any Other Pitch, Multiply by that Pitch.

T.	P. D.	T.	P. D.	T.	P. D.	T.	P. D.
10	3.18	33	10.50	56	17.83	79	25.15
11	3.50	34	10.82	57	18.15	80	25.47
12	3 82	35	11.14	58	18.47	81	25.79
13	4.14	36	11.46	59	18.78	82	26.10
14	4 46	37	11.78	60	19.10	83	26.43
15	4.78	38	12.10	61	19.42	84	26.74
16	5.09	39	12.42	62	19.74	85	27.06
17	5.40	40	12.74	63	20.06	86	27.38
18	5.73	41	13.05	64	20.38	87	27.70
19	6.05	42	13.37	65	20.63	88	28.02
20	6.37	43	13.69	66	21.02	89	28.34
21	6 69	44	14.00	67	21.33	90	28.65
22	7.00	45	14.33	68	21.65	91	28.97
23	7.32	46	14.65	69	21.97	92	29.29
24	7.64	47	14.96	70	22.29	93	29.60
25	7.96	48	15.28	71	22.60	94	29.93
26	8.28	49	15.60	72	22.92	95	30.25
27	8.60	50	15.92	73	23.24	96	30.56
28	8.90	51	16.24	74	23.56	97	30.88
29	9.23	52	16 56	75	23.88	98	31.20
30	9.55	53	16.87	76	24.20	99	31.52
31	9.87	54	17.19	77	24.52	100	31.84
32	10.19	55	17.52	78	24.83		

THE EPICYCLOID.

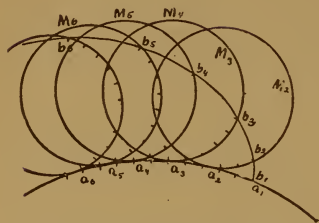## THEORETICAL FORMATION.

FIG. 6   THE EPICYCLOID.

FIG. 7.   EPICYCLOIDAL ENGINE.

The true epicycloid, shown by fig. 6, is perpendicular to the pitch line at the origin a, and forms an endless series of lobes about it, as in fig. 3.

The most convenient and simple process for drawing it, is to step it off with the dividers. Several describing circles, M^1 to M^5, are drawn at random; steps are made, as shown by the figure, from the origin a^1 to past each tangent point, a^1 to a^5, and then the same number back, around each circle, to locate the several points, b^2 to b^5, on the curve, which is then drawn by hand through the points, and is accurately in place if the steps are small.
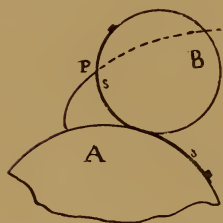
By the mechanical method for drawing the curve, the describing circle, B, is rolled around the pitch circle A, and a tracing point or pencil P, draws the curve. A steel ribbon s, is fastened to the templets at each end, and assists in keeping them in place.

This process is the main principle of the epicycloidal engine, which carries a scribing tool, or a rotary cutter at p, to trace or cut out a templet that is then used in forming gear teeth or gear cutters.

It is, of course, the most accurate method known, but it is not available for ordinary purposes, for unless the templets are well made and skillfully handled, the resulting curve will be poorly drawn, and the method, although simple in principle, may be considered difficult in its practical application.

PRACTICAL FORMATION.

Of course nothing but the perfect curve will answer its purpose with perfect accuracy, but the epicycloid is a peculiar curve which cannot be accurately drawn by any simple process, or with common instruments, particularly when the teeth are small, and it is customary to use arcs of circles or other curves, which approximate as nearly as possible to the true curve.

Such an arc can be made to agree with the curve so closely that it is a needless refinement to be more particular for most practical purposes, such as drafting teeth, making wooden cogs or patterns for cast teeth, or even the templets for shaping gear cutters and planing bevel gear teeth.

Some makers of rough cast or heavy planed gearing go to great expense to construct the (supposed to be) theoretically true epicycloid, by means of rolling circles. This practice looks very much indeed like accuracy, but if he had an absolutely true curve as a templet, supposing he could make such a thing, the maker of this class of work could not produce from it a working tooth more nearly perfect than if the templet was properly constructed of circular arcs. It is labor lost to lay out teeth to the thousandth of an inch, that must be constructed with ordinary hand or machine tools, or shaped with a chisel and mallet.

Furthermore, it is a question if the delicate processes and epicycloidal engines used for the finest cut gear work, can serve practical purposes and construct templets to work from, better than intelligent and skillful hand-work. It is a fact that the best work in this line is made from templets that are laid out by theory, but dressed into shape and perfected by hand and eye processes.

ODONTOGRAPHS.

Many arbitrary or "rule of thumb" methods for shaping gear teeth have been propo ed, but they are generally worthless, and reliance should be placed only on such as are founded on the mathematical principles of the curve to be imitated. Of these only three are known to the writer.
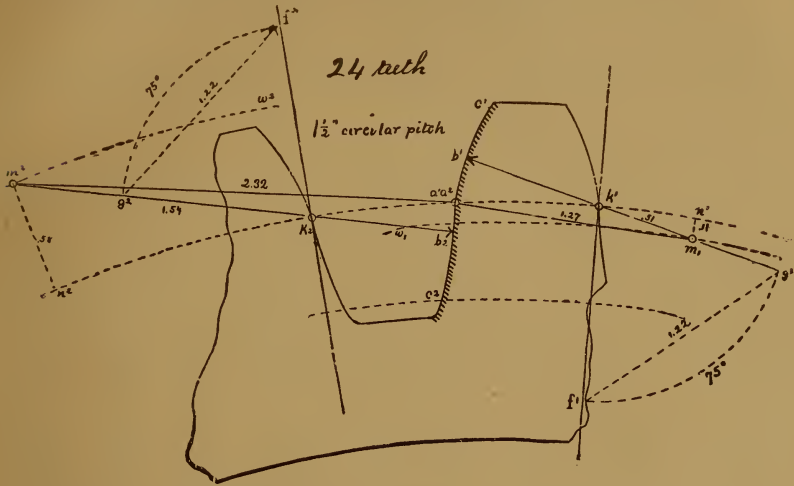
THE WILLIS ODONTOGRAPH is a method for finding the center m of the circle which is tangent to the epicycloid a b c, at the point b, where it is cut by a line b m, which passes through the adjacent pitch point k, and makes the angle g k f=75° with the radial line k f.

The radius used, is not the line m b, but the more convenient line m a.

The instrument is nothing whatever but a piece of card or sheet metal cut to the angle of 75°, which is laid against the radial line k f, as a guide for drawing the line k m. The center distance k m, to be laid off along the line thus drawn is given by a table that accompanies the instrument.

No instrument is necessary, for the line k m may be placed by drawing the arc f g with a radius of one inch, and laying off the chord f g=1.22 inch. The tabular distance k m can be readily computed from

$$k_1 \, m_1 = \frac{c}{2.03} \cdot \frac{t}{t+12}$$

$$k_2 \, m_2 = \frac{c}{2.03} \cdot \frac{t}{t-12}$$

in which c is the circular pitch in inches, and t is the number of teeth in the gear.

The Willis odontograph, as found in use, is confined to the single case of an interchangeable series running from twelve teeth to a rack, but for any possible pair of gears the angle becomes

$$g \, k \, f = 90° - \frac{180°}{s}$$

and

$$k_1 \, m_1 = \frac{s \, c}{6.28} \cdot \frac{t}{t+s} \cdot \sin. \frac{180°}{s}$$

$$k_2 \, m_2 = \frac{s \, c}{6.28} \cdot \frac{t}{t-s} \cdot \sin. \frac{180°}{s}$$

in which t is the number of teeth in the gear being drawn and s the number in the mate.

The accuracy of the Willis circular arc will be examined further on.

THE IMPROVED WILLIS ODONTOGRAPH.
EPICYCLOIDAL TEETH.

| NUMBER OF TEETH IN THE GEAR.		FOR ONE DIAMETRAL PITCH. For any other pitch, divide by that pitch.				FOR ONE INCH CIRCULAR PITCH. For any other pitch, multiply by that pitch.			
		FACES.		FLANKS.		FACES.		FLANKS.	
Exact.	Intervals.	Rad.	Dis.	Rad.	Dis.	Rad.	Dis.	Rad.	Dis.
12	12	2.30	.15	∞	∞	.73	.05	∞	∞
$13\frac{1}{2}$	13–14	2.35	.16	15.42	10.25	.75	.05	4.92	3.26
$15\frac{1}{2}$	15–16	2.40	.17	8.38	3.86	.77	.05	2.66	1.24
$17\frac{1}{2}$	17–18	2.45	.18	6.43	2.35	.78	.06	2.05	.75
20	19–21	2.50	.19	5.38	1.62	.80	.06	1.72	.52
23	22–24	2.55	.21	4.75	1.23	.81	.07	1.52	.39
27	25–29	2.61	.23	4.31	.98	.83	.07	1.36	.31
33	30–36	2.68	.25	3.97	.79	.85	.08	1.26	.26
42	37–48	2.75	.27	3.69	.66	.88	.09	1.18	.21
58	49–72	2.83	.30	3.49	.57	.90	.10	1.10	.18
97	73–144	2.93	.33	3.30	.49	.93	.11	1.05	.15
290	145–rack.	3.04	.37	3.18	.42	.97	.12	1.01	.13

THE IMPROVED WILLIS ODONTOGRAPH.

I have carefully calculated the distances m_1 n_1 and m_2 n_2 of the circles of centers from the pitch line, and also the radii a_1 m_1 and a_2 m_2, and have arranged them in the table above, so that the data resulting from the usual process can be obtained without the usual labor.

This improved Willis process will produce exactly the same circular arc as the usual method, with the same theoretical error, but its operation is simpler and less liable to errors of manipulation.

By the usual process it is necessary to draw two radial lines, and to lay off a line at an angle with each. The tabular distances laid off on these lines, will locate the two centers. The two circles of centers are then drawn through them, and the dividers set to the radii to be used.

By the new process the circles of centers are drawn at once without preliminary constructions, at the tabular distances from the pitch line, and the table also gives the radii to be taken on the dividers. No special instrument is required, no angles or special lines are drawn to locate the centers, and the chance of error is much less.

This process, however, is not as correct, and is no simpler or more convenient than the new odontographic process given further on.
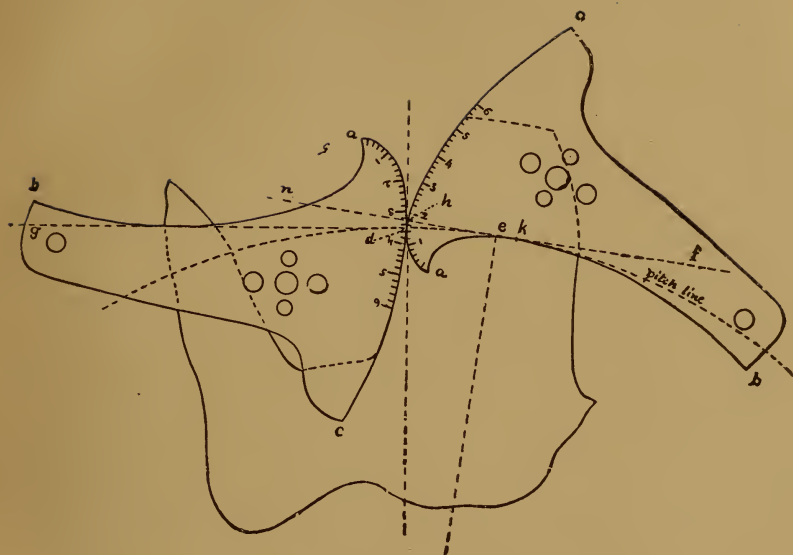
ROBINSON'S TEMPLET ODONTOGRAPH.

This ingenious instrument, the invention of Prof. S. W. Robinson of the Ohio State University at Columbus, is based on the fact that some part of a certain curve of uniformly increasing curvature, called the logarithmic spiral, can be made to agree with the true curve of a gear tooth with a degree of approximation that is very precise.

It is a sheet metal templet having a graduated curved edge a c, shaped to a logarithmic spiral, and a hollow edge a b shaped to its evolute, an equal logarithmic spiral.

To apply the instrument, draw a radial line from the pitch point d on the pitch line, and another from e, the center of the tooth, and then draw tangents d g and n e f, square with the radial lines.

The instrument is then so placed that a certain graduation, given by accompanying tables, is at the point h on the tangent n e f, while the graduated edge a c, is at the pitch point d, and the hollow edge a b, just touches the tangent line n e f at k, and then the face of the tooth is drawn with a pen along the graduated edge. The flank is similarly located by placing the instrument so that a certain other graduation is at the pitch point d, while its hollow edge touches the tangent line g d.

The full theory of this instrument would be out of place here, but may be found in No. 24 of Van Nostrand's Science Series, or in Van Nostrand's Magazine for July, 1876.

A NEW ODONTOGRAPH.

Having frequently to apply the Willis Odontograph, it occurred to me that the process would be much simplified and much time and labor saved if the location of the circles of centers and the lengths of the radii were computed and tabulated, thus forming the improved Willis method already described.

It was then evident that the process would be precisely the same, and the result much improved, if the centers tabulated were the centers of the nearest possible approximating circles, rather than of the Willis circles, and I have embodied this idea in the following tables.

I have carefully computed, by accurate trigonometrical methods, and have tabulated the location of the center of the circular arc that passes through the three most important points on the curve, at the pitch line a, fig. 9, at the addendum line k, and the point c, half way between.

The tables locate this center directly, giving its distance from the pitch line, and from the pitch point.

The circles of centers are drawn at the tabular distances "dis" inside and outside the pitch lines, and all the faces and flanks are drawn from centers on these circles, with the dividers set to the tabular radii "rad."

The tables are arranged in an equidistant series of twelve intervals. For ordinary purposes the tabular value for any interval can be used for any tooth in that interval, but for greater precision it is exact only for the given "exact" number, and intermediate values must be taken for intermediate teeth.

The tables are arranged for both the diametral and circular pitch systems. The former is much the more manageable and should be used when the work is not to interchange with work already made on the latter system.

The first table, giving an interchangeable set, from twelve teeth upwards, is the one for general use.

The second, or radial flank table, is inserted because teeth are sometimes drawn that way, but, as before explained, they are weak, not interchangeable, and but a mere shade more direct in their action than the interchangeable style.

ACCURACY OF THE ODONTOGRAPH.

The assertion is often made that no circular arc can be made to do duty for the epicycloid, but for rough work, but it can be shown that the statement is not true if applied to the new method, for few mechanical processes can be made to work closer to a given example, than this arc is close to the true curve.

Fig. 9.

Figure 9 shows the true curve, and both the new and the Willis approximating arcs, the actual proportions being exagerated to show the errors more clearly.

The Willis arc runs altogether within the true curve, while the new arc crosses it twice.

We will take, for an example, the case of a twelve tooth pinion, which will show the errors at their greatest, and calculate them with great care for a tooth of three inch circular pitch, which is twice the size of the figure on page 13, and may be considered a very large tooth.

The distance from pitch line to addendum line is divided into eight equal spaces by parallel circles, and the distance along each circle, in ten thousandths of an inch, from the true curve to each odontographic arc, is as follows:

	GRANT.	WILLIS.
At a	.0000	.0000 inches
" b	+.0088	+.0175 "
" c	+.0091	+.0244 "
" d	+.0056	+.0283 "
" e	.0000	+.0288 "
" f	−.0036	+.0297 "
" g	−.0061	+.0308 "
" h	−.0046	+.0342 "
" k	.0000	+.0397 "
Average,	.0042	.0260 "

It is seen that the new arc is in no place one hundredth of an inch in error, and that for a tooth of four pitch, a large size for cut work, its average error is one thousandth of an inch. A greater accuracy than this would be of no practical value.

The twelve tooth gear, for which the errors of both arcs were computed, shows them at their maximum value, for, as the number of teeth in the gear increases, the errors diminish, and for several locations their values for the new arc at c, which is the point of greatest error, are as follows:

$$\text{For } t = 12 \qquad c = .009 \text{ inches.}$$
$$\text{" } \quad 20 \qquad \text{" } .008 \text{ "}$$
$$\text{" } \quad 40 \qquad \text{" } .006 \text{ "}$$
$$\text{" } \quad 100 \qquad \text{" } .004 \text{ "}$$
$$\text{" } \quad 300 \qquad \text{" } .002 \text{ "}$$

and the errors of the Willis arc are subject to the same rule.

The error of the Willis arc is plainly shown, at its greatest value, by the figure on page 13, where the dotted faces of the pinion teeth are correctly located by the Willis method.

To further test the accuracy of the new method, construct the same tooth face several times by the same process, using either the method by points, or the usual Willis process. Unless the work is most carefully performed, it will be found that the several results will not agree with each other by amounts that are noticeable, while by the new method they will be substantially the same curve.

The new arc is most nearly correct at the most important point, the upper part of the curve, just where the Willis arc is most out of place, or where the true curve, unless drawn by some delicate and costly apparatus, in most likely to be out of place.

CIRCULAR AND DIAMETRAL PITCHES COMPARED.

CIR. P.	DM. P.
6	.52
5½	.58
5	.63
4½	.70
4	.78
3½	.90
3	1.05
2¾	1.15
2½	1.25
2¼	1.40
2	1.57
1¾	1.80
1½	2.10
1¼	2.50
1	3.14
¾	4.20
½	6.28

DM. P.	CIR. P.
½	6.28
¾	4.20
1	3.14
1¼	2 50
1½	2.10
1¾	1 80
2	1.57
2½	1.25
3	1.05
3½	.90
4	.78
5	.63
6	.52
7	.45
8	.39
9	.35
10	.31

THE NEW ODONTOGRAPH.
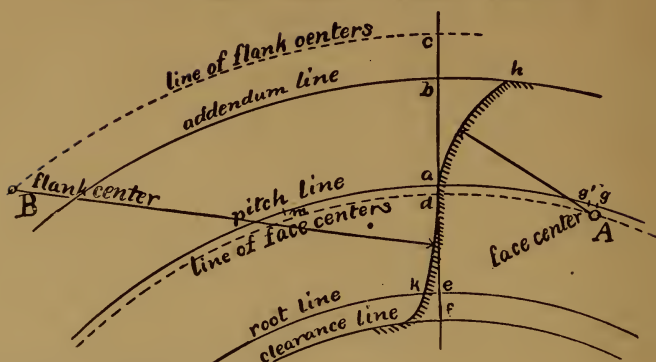
GENERAL DIRECTIONS.

Draw the pitch line and divide it for the pitch points m a g. Take from the tables, multiply or divide, as the case may require, by the pitch in use, and lay off, the addendum a b and a c, the clearance e f, the backlash g g', the face distance a d, and the flank distance a c. Draw the addendum line through b, the root line through e, the clearance line through f, the line of face centers through d, and the line of flank centers through c. Set the dividers to the face radius, and draw all the faces a b from centers A. Set to the flank radius, and draw all the flanks a k from centers B. Round the flanks into the clearance line. The flanks of a gear of twelve teeth are straight radial lines.

ODONTOGRAPH TABLE.

EPICYCLOIDAL TEETH.

INTERCHANGEABLE SERIES.

FROM A PINION OF TWELVE TEETH TO A RACK.

NUMBER OF TEETH IN THE GEAR.		FOR ONE DIAMETRAL PITCH. For any other pitch, divide by that pitch.				FOR ONE INCH CIRCULAR PITCH. For any other pitch, multiply by that pitch.			
		FACES.		FLANKS.		FACES.		FLANKS.	
Exact.	Intervals.	Rad.	Dis.	Rad.	Dis.	Rad.	Dis.	Rad.	Dis.
12	12	2.01	.06	∞	∞	.64	.02	∞	∞
13½	13–14	2.04	.07	15.10	9.43	.65	.02	4.80	3.00
15½	15–16	2.10	.09	7.86	3.46	.67	.03	2.50	1.10
17½	17–18	2.14	.11	6.13	2.20	.68	.04	1.95	.70
20	19–21	2.20	.13	5.12	1.57	.70	.04	1.63	.50
23	22–24	2.26	.15	4.50	1.13	.72	.05	1.43	.36
27	25–29	2.33	.16	4.10	.96	.74	.05	1.30	.29
33	30–36	2.40	.19	3.80	.72	.76	.06	1.20	.23
42	37–48	2 48	.22	3.52	.63	.79	.07	1.12	.20
58	49–72	2 60	.25	3 33	.54	.83	.08	1.06	.17
97	73–144	2.83	.28	3.14	.44	.90	.09	1.00	.14
290	145–rack.	2.92	.31	3.00	.38	.93	.10	.95	.12

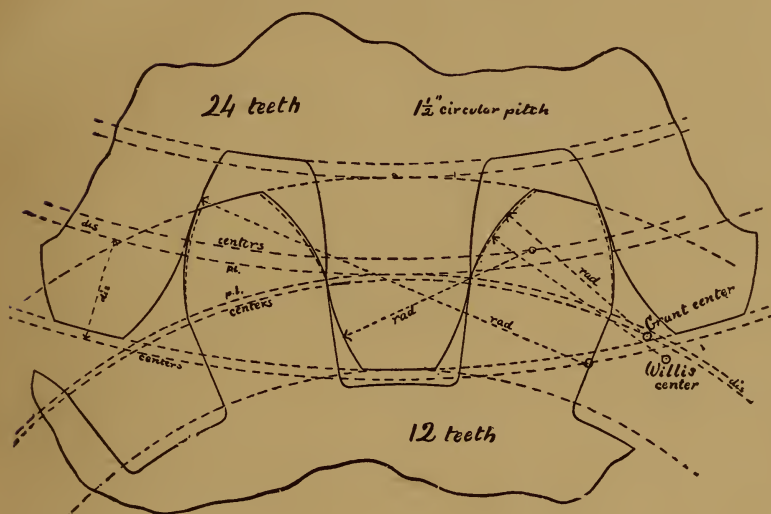# A PRACTICAL EXAMPLE

OF THE WORK OF THE NEW ODONTOGRAPH.

Fig. 10.

INTERCHANGEABLE SERIES.

EXAMPLE.—A gear of 24 teeth, and a gear of 12 teeth, of 1½ circular pitch.

DATA.—Take from the table the numbers to be used, which are as follows when multiplied by 1½.

For 24 teeth, face rad, = 1.08   face dis, = .07.
"   24   "   flank " = 2.15   flank " = .54.
"   12   "   face " = .96   face " = .03.
"   12   "   flank " = ∞   flank " = ∞

Also take from the proper tables the pitch diameters 5.73 and 11.46 inches, the addendum, .5 inch, and clearance, .06 inch.

PROCESS.—Draw the two pitch lines, and divide for the pitch points.   Draw the addendum, root, and clearance lines of both gears.

Draw the circles of centers, .03 inside the pitch line of the 12 tooth gear, and .07 inside of that of the other.   Draw the circles of flank centers, .54 outside the pitch line of the 24 tooth gear, and draw straight radial flanks for the 12 tooth gear.

Draw the faces of the 12 tooth gear with the radius .96, and draw the faces of the 24 tooth gear with the radius, 1.08, and the flanks with the radius 2.15.

Round the flanks into the root line, and allow backlash by thinning the teeth according to judgement.

The dotted faces of the 12 tooth gear show them as they would be laid out by the Willis odontograph, and the figure also shows the two centers

RADIAL FLANK SYSTEM.
TEETH NOT INTERCHANGEABLE.

Gears on this system must work together in pairs, each gear being fitted to its mate and to no other. See page 3. The process is the same that has been described on page 12 for the interchangeable set.
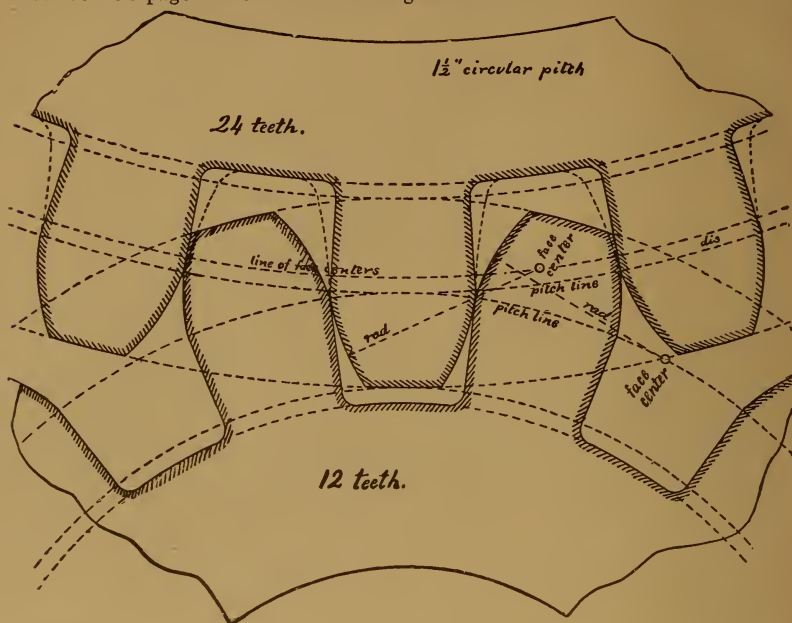

FIG. 11.

RADIAL FLANK SYSTEM.

EXPLANATION OF THE TABLE.—The upper number in each square is the face radius, the lower is the center distance.

The centers are mostly insid the pitch line, but some are on the line, and those having the negative sign are outside of it.

The tabular numbers are for one inch circular pitch, and must be multiplied by any other circular pitch in use. For the value for any diametral pitch, multiply the tabular number by 3.14, and then divide by the diametral pitch in use.

EXAMPLE.—A gear of 12 teeth, paired with a gear of 24 teeth. Circular pitch 1½ inches.

DATA.—Take from the table for 12 teeth into 24, face radius =.68 and center distance = 0, and for 24 teeth into 12. radius = 72, and distance = .05. These multiplied by 1½ give the values for use on the drawing, 12 rad. =1.02, 12 dis = 0, 24 rad. = 1.08, and 24 dis. = .07.

The addendum is one third the pitch, = ½ inch, and the proper tables give the clearance =.06, and the pitch diameters = 5.73 and 11.46 inches.

PROCESS.—Draw the two pitch lines 5.73 and 11.46 inches in diameter and space them for the teeth.

Lay off the addendum, .5 inch, and the clearance. .06 inch, and draw the addendum, root, and clearance lines.

Draw all the faces of the twelve tooth gear, from centers on its pitch line, with the radius 1.02. Draw all the faces of the 24 tooth gear from centers on a line .07 inch inside its pitch line, with the radius 1.08 inches. Draw straight radial lines for the flanks of all the teeth.

ODONTOGRAPH TABLE.

EPICYCLOIDAL TEETH.

RADIAL FLANK TABLE.

FOR ANY POSSIBLE PAIR OF GEARS, NOT INTERCHANGEABLE.

Multiply by the CIRCULAR PITCH.

Divide by the DIAMETRAL PITCH, and then multiply by 3.14.

NUMBER OF TEETH IN GEAR BEING DRAWN.		NUMBER OF TEETH IN THE MATE.											
Exact.	Intervals.	12	13 14	15 16	17 18	19 21	22 24	25 29	30 36	37 48	49 72	73 144	145 rack
12	12	.64 .02	.64 .01	.65 .01	.66 .01	.67 0	.68 0	.69 −.01	.70 −.01	.71 −.02	.73 −.02	.74 −.03	.75 −.03
13½	13–14	.65 .02	.66 .02	.67 .01	.68 .01	.69 .01	.70 0	.72 0	.74 −.01	.75 −.01	.76 −.02	.78 −.02	.79 −.03
15½	15–16	.67 .03	.68 .02	.69 .02	.70 .01	.72 .01	.74 .01	.75 0	.78 0	.79 −.01	.82 −.02	.84 −.02	.84 −.03
17½	17–18	.68 .04	.70 .03	.71 .02	.73 .02	.75 .01	.77 .01	.78 .01	.82 0	.84 −.01	.87 −.01	.89 −.02	.90 −.03
20	19–21	.70 .04	.72 .04	.74 .03	.76 .02	.79 .02	.81 .01	.83 .01	.87 0	.90 0	.93 −.01	.96 −.02	.96 −.03
23	22–24	.72 .05	.74 .04	.76 .04	.79 .03	.82 .02	.84 .02	.87 .01	.91 .01	.94 0	.98 −.01	1.01 −.02	1.03 −.03
27	25–29	.74 .05	.76 .05	.79 .04	.82 .04	.85 .03	.87 .02	.92 .02	.96 .01	.99 0	1.03 −.01	1.07 −.02	1.10 −.03
33	30–36	.76 .06	.79 .05	.83 .05	.86 .04	.90 .03	.94 .03	.98 .02	1.02 .01	1.06 0	1.11 0	1.17 −.01	1.23 −.02
42	37–48	.79 .07	.83 .06	.86 .05	.90 .05	.96 .04	.98 .04	1.03 .03	1.08 .03	1.14 .02	1.20 0	1.25 0	1.37 −.01
58	49–72	.83 .08	.87 .07	.91 .07	.96 .06	1.02 .06	1.05 .05	1.10 .04	1.17 .04	1.24 .03	1.30 .02	1.43 0	1.58 0
97	73–144	.90 .09	.93 .08	.97 .08	1.01 .07	1.07 .07	1.11 .06	1:18 .06	1.28 .05	1.34 .04	1.47 .03	1.65 .02	2.03 0
290	145 rack	.93 .10	.96 .09	1.00 .09	1.05 .09	1.10 .08	1.16 .08	1.24 .07	1.37 .07	1 50 .06	1.70 .04	2.12 .03	2.90 .02

THE INVOLUTE TOOTH.

With the exception of the epicycloid, the only curve in extensive use for the working face of a gear tooth, is the involute.

THE INVOLUTE CURVE.

As the rolling circle A of fig. 3 increases in size, it finally, when of infinite diameter, becomes the straight line d g of fig. 15, while the epicycloid traced by a fixed point in the circle becomes the involute.
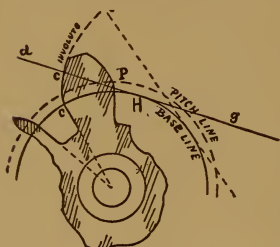

FIG. 15.   THE INVOLUTE.

The involute is, therefore, not a new or separate curve, but simply a particular case of the epicycloid.   It is the infinite form of the epicycloid.*

As the rolling circle of infinite diameter is the same thing as a straight line, the involute can be formed by a fixed tracing point in a cord which is unwound from a circle, called its "base circle," which has been wrapped or "involved" in it, and from this property it derives its name.

ITS UNIFORM ACTION.
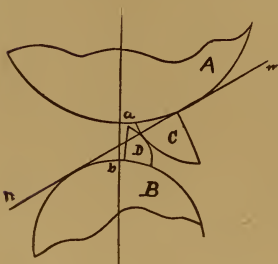
FIG. 16.   EXTERNAL EPICYCLOIDS.

If the two circles A and B, fig. 16, are separated by the distance a b, and work together by means of two external epicycloids C and D, the motion communicated will be irregular, for the conditions of uniformity are that the two circles shall touch, and that the external curve of one shall work with the internal curve of the other.   See page 2 and figure 4.

The amount of this irregularity will depend on the proportion between the separating distance a b and the diameter of the rolling circle which describes the epicycloids.   If the proportion is very small, the irregularity will be very small, and if the rolling circle has an infinitely great diameter, the proportion, and the irregularity will be infinitely small, that is, zero.   Therefore, involutes will work together with perfect regularity and are suitable curves for gear teeth.

ITS ADJUSTIBILITY.

If the rolling circle is infinitely large, the proportion between the separating distance and it will always be zero, and it will not be altered by any finite alteration of the former, and therefore the uniformity of the action of involute teeth is not in any way dependent upon, or affected by any change of the separating distance.   The action will be perfect as long as the curves remain in contact, and this is a property of the greatest practical value, which gives the involute a great advantage over every other known or possible curve.

The curve of any gear tooth must of necessity be a "rolled curve" formed by a fixed object attached to the plane of or moving with some curve that rolls upon the base curve of the tooth, and, as the involute is the infinite form of any rolled curve, it is the only form that can possess this property of adjustibility.

* The exact nature of the involute curve is more fully treated of in a paper in the appendix, on "The Normal Theory of the Gear Tooth Curve."

ITS UNIFORM PRESSURE AND FRICTION.

The point of contact of the two involutes C and D will always be upon the straight line of action m n, the common tangent of the two base circles, commencing at its point of tangency with one circle, and ending at the same point on the other.

The direct pressure between the two teeth will always be in the direction of the line of action, and uniform both in direction and in amount, a property that is peculiar to the involute curve, and which contributes greatly to the smooth action and even wear of involute teeth. Friction is substantially in proportion to direct pressure, and when the pressure is uniform, the friction will be uniform, and no part of the curve will be more likely to wear away than any other part. The durability of a tooth, particularly when doing heavy work, depends on the uniformity of the friction as well as upon its absolute amount.

THEORETICAL CONSTRUCTION.

To draw the involute curve through the pitch point a of two pitch circles A and B, draw the line of action m n at any desired angle with the line of

FIG. 17. CONSTRUCTION.

centers, usually 75°, and then draw the base circles C and D, touching the line of action at e and d, where the perpendicular radial lines e g and f d meet it. From a, step off any number of short steps along the line of action and around the base line to any point s, then draw any number of tangent lines b c, t v, then step off the distances s b c, s t v, sb, etc., each equal to s d a, and the points c, v, b, etc., will be points of the curve. Any line, as w c x drawn through c at right angles to h c, will be tangent to the curve. The working part of the curve must not be extended beyond the circle k e p thro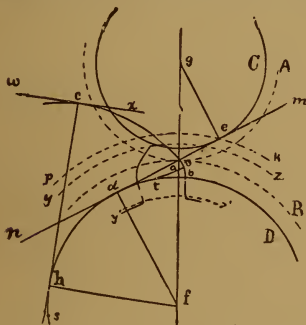ugh the point of contact of the line of action m n and the base line C, for beyond that point it will interfere with the radial flank of the tooth it works with.

The curve is generally limited by the addendum line z y, at an arbitrary distance from the pitch line B, and ends at b on the base line D, where it is perpendicular to the base line. It is continued within the base line by a radial line as far as the root line z y, and is then rounded into the clearance line.

The matter under epicycloidal teeth, pages 3, 4, and 5, regarding the pitch, addendum, clearance, and backlash, will apply as well to involute teeth.

ANGLE OF ACTION.

The angle m a g may be less, but not greater, than the value found from the formula

$$m \, a \, g = 90^\circ - \frac{180^\circ}{s}$$

in which s is the number of teeth in the smallest gear in the pair. If the angle is greater than this the motion will not be continuous, as each pinion tooth will pass out of action before the next one is in position to act.

INTERCHANGEABLE SETS.

Any number of involute gears from base circles of different diameters will work together correctly and interchangeably if all are of the same pitch, and have the same angle of action.

If we put s = 12 teeth, we find

$$m \, a \, g = 90^\circ - \frac{180^\circ}{12} = 75^\circ$$

the value for the common twelve to rack interchangeable set, and if we use fifteen as the smallest number of teeth in the set, we have an angle of action of 78°.

PRACTICAL CONSTRUCTION.

When the involute is to be brought into use, we meet with the same difficulties as with the epicycloid, for its theoretically correct construction is not easily and accurately accomplished, and we must adopt some short cut of approximative accuracy.

The principle of the epicycloidal engine of fig. 7 may be applied to the construction of the involute, the ribbon s being drawn tight and straight as it is unwound from the base circle, but the same difficulties prevent its use for ordinary purposes.

THE OLD RULE.

A defective rule in common use draws the whole curve from base line to addendum line, as one circular arc. The angle m a g is laid off at 75°, sometimes at 75½°, the distance a c is made equal to one quarter of the pitch radius a g, and the tooth curve is drawn from c as a center.
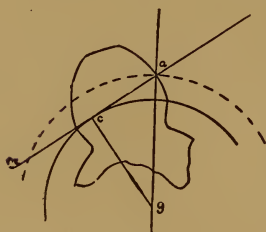
FIG. 18. THE OLD RULE.

This rule is simple, to be sure, but it gives the faces shown by the dotted lines of the figure on page 23, and is abominably wrong and worthless.

If it would round off the points of the teeth of a large gear, it would be useful to correct interference, but it greatly rounds the teeth of a small gear that needs little or no correction, and gives the curve on a large gear in nearly its theoretical position, without the allowance for interference that must be made.

It is not to be wondered that the involute tooth is in small favor with practical mechanics who use this bungling method, and who do not understand that the trouble is not in the involute system, but in its defective application.

A NEW METHOD.

In devising a method for drafting the involute tooth, I have borne in mind that a minute degree of accuracy is not the essential requirement, for although substantial accuracy must be secured, simplicity and convenience are qualities that must also be considered.
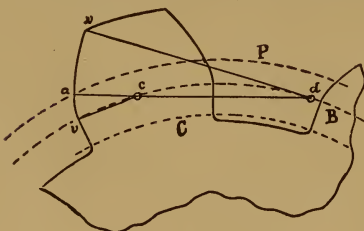

FIG. 19. THE NEW METHOD.

The method, in general terms, and given in full on pages 22 and 23, is to give, by a table, the distance of the base circle B, see fig. 19, inside the pitch circle P, and to give by the same table, the distances or radii a c and a d from the pitch point a to centers c and d on the base line. The face arc a w is drawn from the center d and the flank arc a v from the center c.

The table, page 22, is for one diametral pitch, and covers the common twelve to rack interchangeable set.

INTERFERENCE.

As indicated above, the involute face will interfere with the radial flank of the mating tooth if the addendum is greater than a certain amount, and as the addendum in common use for the interchangeable set generally exceeds this limit, we must gererally make corrections to avoid this trouble.

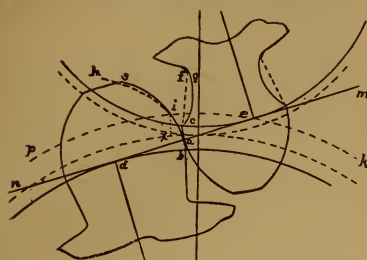Figure 20 shows the interference, its effect, and its correction.

FIG. 20. INTERFERENCE.

The working face of the involute should be limited at i by the circle k p through the tangent point e, but if the usual addendum continues it beyond that line, to s, the extension s i will interfere with the radial flank c f, and the uniformity of the action will be destroyed.

To correct it we must either weaken and spoil the shape of the mate tooth by undercutting the flank c f by an epitrochoidal line c g, or we may, and much better, round off the point of the tooth by an epicycloidal curve i h.

The amount of this interference will depend on, and increases with, the angle of action, and also depends upon the number of teeth in each gear. It is greatest on a large gear or rack that runs in a small pinion, and least on a pinion running in a large gear. When the angle of action is 75° there is no interference when both gears of a pair have thirty or more teeth, or when an equal pair have twenty-one or more teeth. When one gear has more, and the other has less than thirty teeth, the larger may need correction, but the smaller never will.

The amount of the interference, the correction to be made by rounding off the point of the tooth, is very small and may generally be neglected on small pinions.

It is given by the lower figures in the table, which shows that it is never more than a sixteenth of an inch on a large tooth of one diametral, or three inch circular pitch, and not over two or three hundredths of an inch on a gear of that pitch having few teeth. The table also shows by the upper figures the limit point or distance i x above the pitch line where the interference commences.

The tabular numbers must be divided by the diametral pitch that may be in use, and for any circular pitch it is sufficient to divide the tabular number by 3 and then multiply by the pitch.

The table takes no notice of an interference of less than a hundredth of an inch on a tooth of three inch circular pitch.

When, as is usually and should always be the case, the gear being drawn belongs to the twelve to rack interchangeable set, the interference should be computed for a mate gear of twelve teeth, or by the first vertical column of the table. In this case the error will not be perceptible if the limit distance to point of first interference be always assumed to be half the addendum.

When the work is upon a rough cog wheel or mill gear, or upon a pattern for a cast gear, the only correction needed for interference, is a slight rounding off of the points if it is a rack or very large gear, and a mere touch on the point of a gear of few teeth.

Interference Table

For one diametral pitch and 3 1-7 inch circular pitch. Angle of action, 75°.

Number of Teeth in the Mate.

Number of Teeth in the Gear Being Drawn.	12 / 13	14 / 15	16 / 17	18 / 19	21
12	.58 / .01	.67 / .01			
13–14	.56 / .02	.66 / .01			
15–16	.54 / .02	.64 / .01	.75 / .01		
17–18	.53 / .02	.62 / .02	.72 / .01		
19–21	.51 / .02	.60 / .02	.69 / .01		
22–24	.50 / .02	.58 / .02	.67 / .01		
25–29	.49 / .03	.57 / .02	.65 / .02	.75 / .01	
30–36	.47 / .03	.55 / .02	.63 / .02	.72 / .01	
37–48	.45 / .03	.53 / .02	.61 / .02	.69 / .01	
49–72	.44 / .04	.52 / .03	.59 / .02	.66 / .02	.73 / .01
73–144	.42 / .05	.49 / .04	.56 / .03	.63 / .02	.70 / .01
145–∞	.40 / .06	.46 / .05	.53 / .04	.60 / .02	.67 / .01

EPICYCLOIDAL vs. INVOLUTE TEETH.

A COMPARISON.

The epicycloidal tooth is in much greater use and favor than the involute form, particularly for heavy work, both writers and mechanics generally preferring it, and seldom giving the preference to its rival. It is difficult to account for this favor except, as in the case of the circular pitch system, on the ground that the epicycloid was adopted in the infancy of mechanical science, and holds its place by virtue of prior possession, for the involute has certainly the advantage from every practical point of view.

Space will not permit an extended discussion with the necessarily bulky demonstrations, but, if the two curves be closely and carefully examined under the same conditions within the limits of either the twelve tooth or the fifteen or higher tooth interchangeable series, with the customary addendum, which limitation will cover nine-tenths of the gears in actual use, it will be found that they compare as follows:

I. Adjustibility. Involute teeth alone can possess the remarkable and practically invaluable property, that they are not confined to any fixed radial position with respect to each other, for, as long as one pair of teeth remains in action until the next pair is in position, the perfect uniformity of the action of the curve is not impaired.

The shafts may be at the proper distance apart, or not, as happens, and they may change position by wearing, or variably as when used on rolls, or may be forced together to abolish backlash, and, in fact, the curve is wonderfully adapted to the variable demands, and will accommodate itself to errors and defects that cannot be avoided in practice.

Epicycloidal teeth must be put exactly in place and kept there, and the least variation in position, from bad workmanship in mounting, or by wear or alteration of the bearings in use, will destroy the uniformity of the motion they transmit. When perfectly mounted and carefully kept in order, epicycloidal teeth are as good as any in this respect, but for most practical purposes they are decidedly inferior.

This virtue of the involute is always recognized by writers, but is seldom given the position its importance demands, for it is only as a result of experience in making and using gears, that its importance can be seen at its full value.

II. Uniformity. The direct force exerted by involute teeth on each other, is exactly uniform, both in direction and in amount, and this property ensures *a uniform wearing action of the teeth*, a nearly uniform thrust on the shaft bearings, and a steadiness and smoothness of action that cannot be claimed for epicycloidal teeth under any circumstances.

The direct pressure acting between epicycloidal teeth is variable in amount and very variable in direction, and consequently the friction and wearing action between the teeth, as well as the thrust on the bearings, is variable between wide limits.

III. Friction. The measure, for purposes of comparison, of the loss of power by friction, is the product of the direct pressure between the teeth, multiplied by their rate of sliding motion on each other.

This measure is always in favor of the involute by a decided advantage, although the advantage is usually claimed for the epicycloid, both as to maximum values and average values, and as this is an important point, it should have great weight in deciding between the two forms of teeth, for the element of friction is of chief importance in determining the life of a gear in continual and heavy service.

The epicycloid is mostly in use for heavy gearing from a mistaken view of this point, it being generally taught that its friction is the least.

IV. Thrust on Bearings. Here the advantage is with the epicycloidal tooth, but not by a large amount, and not in a matter of first consequence.

The thrust on the bearings due to the action of the teeth on each other is but a fraction of the whole thrust due to the power being carried, and as

the average thrust of the teeth is but little in favor of the epicycloid, and as the maximum thrust is always from that form of tooth, the two forms may be said to be well balanced in this respect. Moreover, the thrust of the involute is but slightly variable, while that of the epicycloid varies from large values at the points of first and final action to nothing at all at the line of centers, and must give rise to a rattling and uneven action.

V. STRENGTH. The weakest part of a tooth is at its root, and as the involute tooth spreads more than the epicycloidal tooth, it is stronger at that point and has a considerable advantage.

VI. APPEARANCE. This is a small point and a matter of opinion, but is worth mention. The involute is a simple and graceful single curve, while the epicycloid is a double and not mechanically a neat curve, and, *as generally drawn*, has a decided bulge or even a plain corner where the two halves join at the pitch line.

IN GENERAL. As the involute has the advantage of the epicycloid, in nine actual cases out of ten, with respect to adjustibility in position, in uniformity of wear and action, in loss of power and change of shape by friction, in strength, and in appearance, and is but a shade, if any, inferior with regard to the thrust on the bearings, it may be, and should be accorded first place for any and every practical purpose. The writer can imagine no possible case, unless it be in connection with a pinion of very few teeth, where the epicycloid would have either a theoretical or a practical advantage over the involute.

ODONTOGRAPH TABLE.

INVOLUTE TEETH.

Corrected for Interference.

Interchangeable Set.

TEETH.	DIVIDE BY THE DIAMETRAL PITCH.		MULTIPLY BY THE CIRCULAR PITCH.	
	Face Radius.	Flank Radius.	Face Radius.	Flank Radius.
12	2.70	.83	.86	.27
13	2.87	.93	.91	.30
14	3.00	1.02	.95	.33
15	3.15	1.12	1.00	.36
16	3.29	1.22	1.05	.40
17	3.45	1.31	1.09	.43
18	3.59	1.41	1.14	.46
19	3.71	1.53	1.18	.50
20	3.86	1.62	1.22	.53
21	4.00	1.73	1.27	.57
22	4.14	1.83	1.32	.60
23	4.27	1.94	1.36	.63
25	4.56	2.15	1.45	.70
28	4.82	2.37	1.54	.77
31	5.23	2.69	1.67	.88
34	5 77	3.13	1.84	1.00
38	6.30	3.58	2.01	1.16
44	7.08	4.27	2.26	1.38
52	8.13	5.20	2.59	1.70
64	9.68	6.64	3.09	2.18
83	12.11	8.93	3.87	2.90
115	16.18	12.80	5.16	4.15
200	25.86	22.30	8.26	7.30

For intermediate teeth use proportionally intermediate values when great accuracy is desired, but for drafting purposes use the nearest value, thus : — 35 is at one-quarter of the distance from 34 to 38, and the proper values for accurate work are : face radius, 5.90 inches, and flank radius 3.24 inches.

The table is not carried beyond 200 teeth, as the higher numbers are rarely used and the radii are then very great. For drafting purposes use values for 200 teeth for all higher numbers.

The base distance, the distance from pitch line to base line, is always one-sixtieth of the pitch diameter.

SPECIAL PROCESS FOR RACK TEETH.

See the cut on the opposite page.

The flank of the tooth and one-half of the face is a straight line at an angle of 75 degrees, five-sixths of a right angle, with the pitch line.

Draw the outer half of the face of the tooth, one-quarter of its whole length, as a circular arc from a center on the pitch line and with a radius of

2.10 inches divided by the diametral pitch.

.67 inches multiplied by the circular pitch.

The point must be rounded over in this way to avoid interference, if the rack is to mesh with any pinion having less than 28 teeth.

A PRACTICAL EXAMPLE.
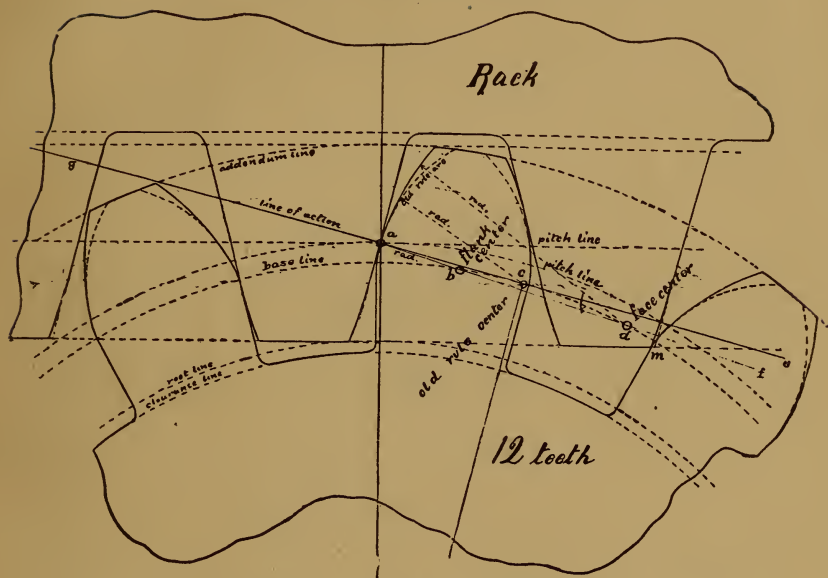
INVOLUTE TEETH.

INTERCHANGEABLE SET.

EXAMPLE. — A rack, and a pinion of twelve teeth, of two diametral pitch.

PINION. — From the tables we have, after dividing by 2, the face radius 1.35 inches, flank radius .42 inches, and clearance .06 inches, The pitch diameter is 6 inches, and the addendum is .5 inches. The base distance, one-sixtieth of the pitch diameter, is .10 inches.

Draw the pitch line and divide it for the pitch points, allowing for backlash if required. Lay off the addendum and the clearance, and draw the addendum line, root line, and clearance line.

Draw the base line .10 inches inside the pitch line.

With the face radius, 1.35 inches, and from centers d on the base line, draw all the face curves from addendum line to pitch line. With the flank radius, .42 inches, and from centers b on the base line, draw all the flanks from the pitch line to the base line.

The flanks inside the base line are straight radial lines.

For fifty or more teeth draw the flank curve from pitch line to root line.

RACK. — Draw by the special rule, the radius for the point being 1.05 inches.

NOTE. — The dotted lines on the pinion teeth show the work of the common rule for involute teeth, as explained on page 18 and given by most of the "gear charts" and works on practical mechanism. The same rule draws the rack tooth with a point that is not rounded. The "old rule" is as worthless as it is simple.
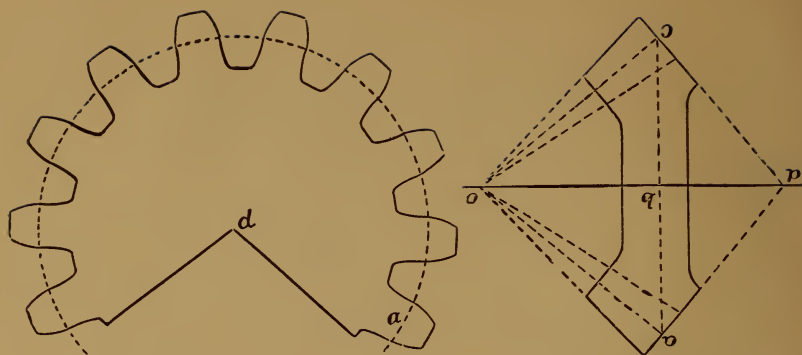
BEVEL GEARS.

In laying out the teeth of a bevel gear but one new point needs to be considered. The working pitch diameter a b c is not to be used, but the teeth are to be drawn on the conical pitch diameter a d c, developed or rolled out as in fig. 25.

The conical diameter a d c may be found from a drawing, or if the gears are of some common proportion, from the following table by multiplying the true pitch diameters by the tabular numbers given for that proportion.

TABLE OF CONICAL PITCH DIAMETERS OF BEVEL GEARS.

Proportion.	Larger Gear.	Smaller Gear.
1 to 1	1.41	1.41
2 " 1	2.24	1.12
3 " 2	1.80	1.20
3 " 1	3.16	1.05
4 " 3	1.67	1.25
4 " 1	4.12	1.03
5 " 4	1.60	1.28
5 " 3	1.94	1.17
5 " 2	2.69	1.08
5 " 1	5.10	1.02
6 " 5	1.56	1.30
6 " 1	6.08	1.01
7 " 1	7.07	1.01
8 " 1	8.06	1.01
9 " 1	9.06	1.01
10 " 1	10.05	1.01

EXAMPLES.—A miter gear, proportion 1 to 1, of 4 pitch, 6″ diameter, and 24 teeth, has a conical diameter of 6″ x 1.41 = 8.46″, and there are 24 x 1.41 = 33.8 teeth on the full circle of the developed cone.

A pair of bevel gears of 3 to 1 proportion, 48″ and 16″ diameters, 36 and 12 teeth, have conical diameters 48″ x 3.16 = 151.68″, and 16″ x 1.05 = 16.80″, and there are 36 x 3.16 = 113.76, and 12 x 1.05 = 12.60 teeth on the full circles of the developed cones.
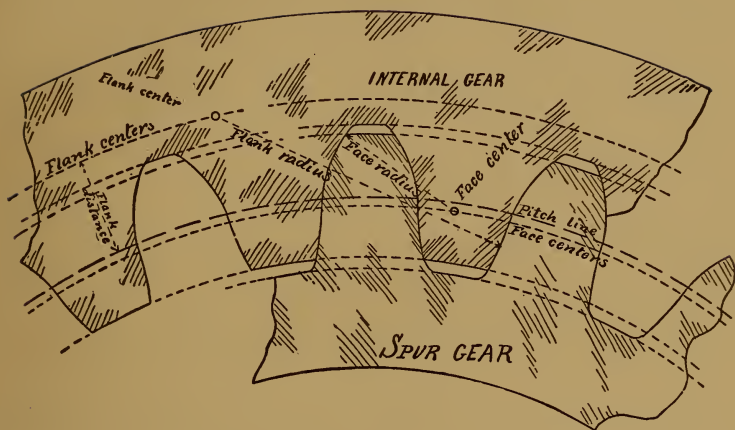
INTERNAL GEARS.

The internal gear, sometimes called the "annular" gear, is drawn by the rules for spur gears, the teeth of a spur gear being the spaces between the teeth of an internal gear of the same pitch diameter, with the backlash and clearance reversed in position.

Involute teeth should end at the base line, the radial part of the flank being omitted, or well rounded over if it is desirable to preserve the appearance of the full tooth.

Internal teeth will interfere, even if properly drawn, unless the gear is considerably larger than the pinion running in it. If drawn for the common twelve to rack interchangeable set, there should be at least twelve more teeth in the gear than in the pinion, and if the difference is less, the teeth must be "doctored" or rounded over until they will pass, and there must be a difference of two teeth in any case.

Involute teeth have a decided advantage over epicycloidal teeth for internal gearing, their action being much more direct, with less sliding and friction.

STRENGTH AND HORSE-POWER OF CAST GEARS.

There are about as many different rules for this purpose, and contradictory results, as there are writers upon the subject. I have preferred not to discuss the theory, but to adopt without question the method given by Thomas Box in his Practical Treatise on Mill Gearing, because that engineer has most carefully considered the practical points in view, and because his formulæ agree almost exactly with a great many cases in actual practice.

STRENGTH OF A TOOTH. — For worm gears, crane gears, and slow-moving gears in general, we have to consider only the dead weight that the tooth can lift with safety.

If we allow the iron to be subjected to but one tenth of its breaking strain, we can use the formula: —

$$W = 350 \, c \, f,$$

in which W is the dead weight to be lifted, c is the circular pitch, and f the face, both in inches.

For the wooden cogs of mortise wheels, use 120 instead of 350 as a factor in the formula.

When the pinion is large enough to insure that two teeth shall always be in fair contact, the load, as found by this rule, may be doubled.

EXAMPLE. — A cast-iron gear of 3'' circular pitch and 6'' face will lift

$$W = 350 \times 3 \times 6 = 6300 \text{ lbs.}$$

HORSE-POWER OF A GEAR. — For very low speeds we can use the formula,

$$\text{HP for low speed} = .0037 \, d \, n \, c \, f,$$

in which d is the pitch diameter, c the circular pitch, and f the face, all in inches, and n is the number of revolutions per minute.

EXAMPLE. — The horse-power of a gear of three feet diameter, three inch pitch, and ten inch face, at eight revolutions per minute, is,

$$\text{HP} = .0037 \times 36 \times 8 \times 3 \times 10 = 32.$$

For ordinary or high speeds, where impact has to be considered, it is found that the above formula gives too high results, and we must use the formula,

$$\text{HP at ordinary speeds} = .012 \, c^2 \, f \, \sqrt{dn.}$$

EXAMPLE. — A gear of three feet diameter, three inch pitch and ten inch face, at one hundred revolutions per minute, will carry but

$$\text{HP} = .012 \times 9 \times 10 \times \sqrt{100 \times 36} = 65 \text{ horse-power,}$$

instead of the 400 horse-power found by the rule for low speeds.

At ordinary or high speeds a wooden cog, on account of its elasticity, will carry as much as or more power than a cast-iron tooth, and we can use .014 instead of .012 in the formula.

When in doubt as to whether a given speed is to be considered high or low, compute the horse-power by both formulæ, and use the smallest result.

For bevel gears the same rules will apply, if we use the pitch diameter and the pitch at the center of the face.

Some rules in use take no account of the face of the gear, but assume that the tooth should be able to bear the whole strain upon one corner.

A tooth that does not bear substantially along its whole face, at several points at least, is a very poor piece of work, and it would be better to straighten the tooth than to force the rule to follow it.

HORSE POWER OF CUT GEARS.

The rules given above for the horse power of gears apply only to gears with rough cast teeth; and in applying them we must consider the speed of the gear as well as its real strength.

One of the chief sources of weakness in a cast gear, is that the continual pounding of the teeth on each other crystalizes the metal so that its strength is gone long before it is worn out.

There are no recorded tests on the horse power of cut gears, but it is generally agreed among those not personally interested in the sale of cast gearing, that a cut gear is much more durable, and that it will carry more power than a cast gear, with the same factor of safety.

In the absence of experimental data, we can only proceed by judgment and inference. It is well settled that the continual pounding of cast gearing is a source of weakness that must be allowed for, and it may be assumed that that source is avoided in the use of cut gears having a smooth and even action.

Until practical tests have been made we can consider that the rule that applies to cast gears for slow speeds where impact need not be considered, can safely be applied at higher speeds to cut gears where there is no impact to be allowed for; and we have the formula : —

Horse power of cut gears at ordinary speeds = .0037 dncf.

Applying this formula to the case of a gear of 36 inches diameter and 3 inch circular pitch, at 100 revolutions per minute, it is found that the cut gear will safely carry six times the power that can be trusted to the cast gear.

But it must be admitted that all that is known concerning the real horse power of a cut gear is a matter of inference, and it is to be hoped that the growing use of cut gearing for conveying heavy powers will furnish data of a more practical and trustworthy nature. Until such data is at hand it may safely be assumed that a cut gear has from two to three times the carrying power of a rough cast gear of the same size.

CONFUSION OF RULES.

The disagreement of standard authorities and the thorough confusion of rules on this subject, is well shown in an interesting paper by J. H. Cooper, in the Journal of the Franklin Institute for July, 1879, in which that engineer has industriously collected twenty-four formulæ from Tredgold, Buchanan, Fairbairn, Box, Molesworth, Haswell, Nystrom and others, and applied them to the practical case of a gear of 60 inches diameter, 7½ inches face, and 3 inches pitch, at 60 revolutions per minute. Cooper found twenty-two different results for this one example, as follows : — 46.31, 47.06, 50.27, 53.18, 56.09, 56.55, 63.62, 66.17, 66.27, 67.96, 68.56, 73.49, 80.78, 84.37, 86.75, 86.80, 86.96, 138.23, 147.27, 163.00, 294.53, and 295.59. Here is variety to suit all tastes, and if a gear is not strong enough for a given purpose according to Fairbairn, it will certainly fill the bill according to Haswell. Diligent enquiry by myself among the cast gear makers of the United States gave the same result as to variety and confusion. I could get little but opinions that were not founded on experiment, and the opinions were of the most indefinite and unsatisfactory character.

All cast gear makers are agreed that a cast gear is more durable than a cut gear, and all cut gear makers are equally certain that a cut gear is more durable than a cast gear.

The stock argument of the makers of cast gearing is that the one-hundredth of an inch thickness of hard scale on a cast tooth makes it more durable than a cut gear from which the scale has been removed. But, from that point of view, they find it very hard to explain why a mortise gear, with soft hickory cogs, is quite as durable as a cast gear with hard teeth.

CHART AND TABLES

FOR

BEVEL GEARS.

A NEW, SIMPLE, AND ACCURATE METHOD FOR FINDING THE ANGLES
AND DIAMETERS OF ANY PAIR OF BEVEL GEARS BY SIMPLE
CALCULATION, AND WITHOUT DRAFTING INSTRU-
MENTS OR SPECIAL TOOLS.

Not one machinist in a dozen will admit that he does not know how to
properly shape a bevel gear blank, but when put to the test, not one in a
score can do it well without an amount of fussing with drafting instru-
ments, and a deal of studying and figuring that looks ridiculous to one who
has studied the subject and knows how simple it really is when it is once
thoroughly understood.

The average bevel gear blank can be relied upon to be wrong in its face
angle, or its outside diameter, or both, even when it has been shaped by a
competent and intelligent general workman, and the simple explanation is
that the only reliance is generally a hurried and poorly made drawing from
which the angles and diameters must be found by measurement, and used
with many chances of error.

This method proceeds by simple calculation, avoiding the use of drafting
instruments, and it will be found to be not only much more accurate, but at
the same time much easier and quicker than any other method. The work-
man who can remember the numbers 1.41 and 81 and the angle 45° needs
no further assistance on miter gears, and on other proportions needs the
table only to supply equally simple data, while two to five minutes is suffi-
cient time for any set of calculations after the method has been learned.

This matter is of more importance than is generally supposed, for bevel
gears unlike spur gears must be exactly correct in diameters and angles,
or no amount of perfection in the cutter or care in the cutting will prevent
a botch.

To be learned this Chart must be studied. If it is not worth
while to give it two or three hours of careful attention it is not worth while
to keep it at all. It is simple and easy to learn but it cannot be taken in at
a glance, or comprehended in ten minutes.

EXPLANATION OF THE METHOD.
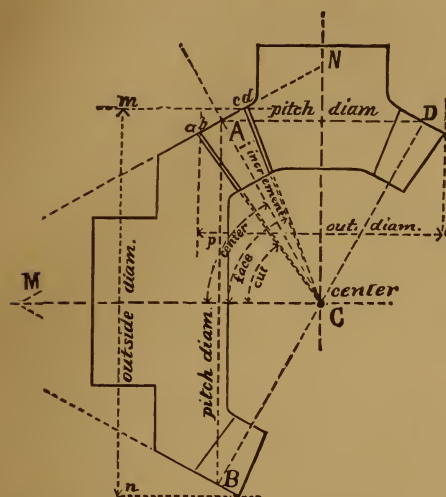
The measures that must be given in advance are the pitch diameters A B and A D, and the numbers of teeth or the diametral pitch, and the measures that must be determined before the blank can be shaped are,

1st, the **outside diameters** m n and p q, each equal to the pitch diameter plus a small increment.

2d, the **center angles** A C M and A C N.

3d, the **face angles** c C M and b C N, each equal to the center angle plus a small increment.

4th, the **cutting angles** a C M and d C N, each equal to the center angle minus a small increment.

Of course all these can be found by first making a drawing and then measuring the diameters and angles, but, although the process is simple enough, and perhaps preferable in some cases in the hands of a draftsman who has the proper tools to use and the skill and knowledge to use them properly; still it is not well adapted to ready and general use in the shop.

Unless made with good instruments that are handled with great care, a drawing is not accurate enough for the purpose, and its results, particularly as to angles, are not apt to be well carried out on the work. It is easy enough to find the proper face angle by a drawing, but not as easy to measure that angle and transfer it to the iron blank.

This method is entirely one of simple calculation, with no instrument but the pen, and no tools to apply its results in the shop but the ordinary scratching dividers and the scale. It is not only more accurate, but, after a few hours work with the table on various examples, it will be easier to work and quicker than the graphical method.

PROCESS IN DETAIL.

First find the **proportion** of the gears by dividing the diameter of the larger gear by that of the smaller, and then use the values in the table opposite that proportion.

The **diameter increments** are found by dividing the tabular increments by the pitch, and the **outside diameters** are found by adding the increments to the pitch diameters.

The **angle increment** is found by dividing the tabular increment by the number of teeth in the larger of the two gears. The **face angle** is the center angle plus the angle increment, and the **cutting angle** is the center angle minus one and one-sixth of the increment.

EXAMPLE.—Given pitch diameters 6″ and 4″, pitch 8, teeth 48 and 32. The proportion is 6 to 4, or 3 to 2, or 1.5 to 1, and the table gives at 1.5 the center angles 56.3° and 33.7°, and the angle increments $\frac{25}{48} = 2°$, and $1\frac{1}{6}$ of 2° $= 2.3°$, from which we find the face angles 56.3° + 2° = 58.3°, and 33.7° + 2° $= 35.7°$, and the cutting angles 56.3° — 2.3° = 54°, and 33.7° — 2.3° = 31.4°.

The diameter increments are $\dfrac{1.11}{8} = .14$, and $\dfrac{1.66}{8} = .21$, and from these we find the outside diameters 6. + .14 = 6.14″, and 4. + .21 = 4.21″.
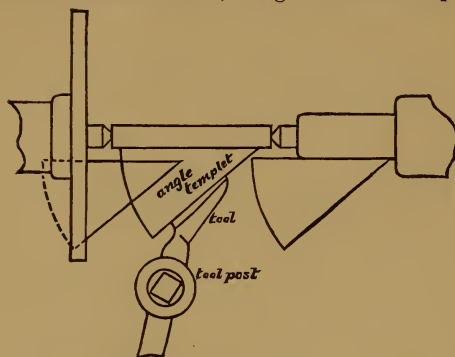
When the proportion falls between two tabular proportions we must use tabular angles and increments that are proportionally between the tabular values.

All diameters should be figured to the nearest hundredth of an inch, and all angles to the nearest tenth of a degree.

PRACTICAL APPLICATION IN THE SHOP.

The best tool for shaping a bevel gear is a compound and graduated slide rest which can be set directly to the angles, but as such an appliance is seldom to be found, and then seldom in good order, the work must generally be done by the use of angle templets.

To use the angle templet, lay one edge against a straight shaft or mandrel held between centers, or against the tail spindle, and adjust a side tool or scraper to the other edge. If the back of the templet be cut square with one edge it can be used by placing it against the face plate.

Be careful with the back angle, for the back of the tooth is not square with its face, and if it is turned square or in any way out of truth, the corners of the teeth will not fit true in the spaces of the mate gear, and the appearance of the job will be spoiled. The true back angle of each gear is the center angle of the other gear of the pair.

SHAFTS NOT AT RIGHT ANGLES.

When, as sometimes happens, the shafts are not at right angles, a simple preliminary drawing must be made. Lay off the shaft angle by means of the table of chords, draw two lines parallel to the shafts at the distances of the half pitch diameters, and from their intersection draw a conical line to the center.

Measure the center angle between either shaft and the conical line, and the proper increments will be found in the table at that center angle. If the center angle is less than 45°, the tabular angle increment must be divided by the number of teeth in the gear, as usual, and then divided by the tabular proportion.

EXAMPLE.—It is found that the center angle of a gear of 8 pitch and 40 teeth is 35°, and this in the table gives the

angle increment $\dfrac{94}{1.43 \times 40} = 1.6°$, and the diameter increment $\dfrac{1.63}{8} = .20$

Proceed separately with the other gear of the pair by measuring and using its center angle.

ERRORS IN DIAMETER.

It will often happen that the outside diameter will be turned too small, or that a casting will not quite turn to the desired size. In this case the diameter should be left as large as possible, and then the other, or mate gear, should be turned under size, *to keep the correct proportion between the pitch diameters.*

For example, if the smaller gear of a pair that are proportioned two to one is found to be $\frac{1}{32}$ inch under size, the larger gear must be turned twice as much, or $\frac{1}{16}$ inch under size.

Great care must be taken to have the smaller gear of an unequal pair very near to size, for any inaccuracy can be balanced only by a proportionally larger inaccuracy of the mate gear. If the smaller gear of a pair that are proportioned six to one is as much as $\frac{1}{32}$ of an inch under size, the larger gear must be turned $\frac{6}{32}$, or $\frac{3}{16}$ inch under size, and this is enough to change the number of teeth the proportion and the face, and spoil the work. The only remedy in such a case is to cut shallow teeth on the pinion, so that its pitch diameter is unchanged.

ANGLE TEMPLETS.

To make an angle templet by the use of the table of chords, draw an arc a d on paper or sheet metal with the dividers set to six inches. Then set the dividers to the chord of the angle and lay it off on the arc, as at b c. Cut to the lines b o and c o.

Similarly, an angle drawn on paper can be measured by drawing an arc across it at a radius of 6″, measuring the chord, and comparing with the table.

TABLE OF CHORDS OF ANGLES,
AT RADIUS OF SIX INCHES.

Degrees	Chord.	Tenths.	Degrees	Chord.	Tenths.	Degrees	Chord.	Tenths.
1	.10		31	3.20		61	6.10	
2	.20		32	3.31		62	6.19	
3	.31		33	3.41		63	6.28	
4	.42		34	3.51		64	6.36	
5	.52		35	3.61		65	6.45	
6	.62		36	3.71		66	6.54	
7	.73		37	3.81		67	6.62	
8	.84		38	3.91		68	6.71	
9	.94		39	4.01		69	6.80	
10	1.04		40	4.10		70	6.89	
11	1.15		41	4.20		71	6.97	
12	1.26		42	4.30		72	7.06	
13	1.36		43	4.40		73	7.14	
14	1.46	.1—.01	44	4.50	.1—.01	74	7.22	.1—.01
15	1.57	.3—.02 .2—.03	45	4.60	.2—.02 .3—.03	75	7.31	.2—.02 .3—.02
16	1.67	.4—.04	46	4.69	.4—.04	76	7.39	.4—.03
17	1.77	.5—.05	47	4.79	.5—.05	77	7.47	.5—.04
18	1.87	.6—.06	48	4.88	.6—.05	78	7.55	.6—.05
19	1.98	.7—.07	49	4.98	.7—.06	79	7.63	.7—.06
20	2.08	.8—.08	50	5.08	.8—.07	80	7.71	.8—.06
21	2.18	.9—.09	51	5.17	.9—.08	81	7.79	.9—.07
22	2.29		52	5.26		82	7.87	
23	2.39		53	5.35		83	7.95	
24	2.49		54	5.45		84	8.03	
25	2.59		55	5.54		85	8.11	
26	2.70		56	5.63		86	8.18	
27	2.80		57	5.72		87	8.26	
28	2.90		58	5.82		88	8.34	
29	3.00		59	5.91		89	8.41	
30	3.10		60	6.00		90	8.48	

The table gives the length of the chord at six inches from center, for any degree. For tenths of a degree, add the value in the small table of the same column.

TABLE OF
INCREMENTS AND ANGLES
FOR
BEVEL GEARS.

Proportion.		Diameter Increment. Divide by pitch.		Angle Increment Divide by number of teeth in larger gear	Center Angles.	
		Larger Gear.	Smaller Gear.		Larger Gear.	Smaller Gear.
1.00	1–1	1.41	1.41	81	45.0	45.0
1.05	——	1.37	1.42	84	46.4	43.6
1.10	——	1.35	1.44	86	47.7	42.3
1.11	10–9	1.34	1.46	87	48.0	42.0
1.13	9–8	1.33	1.47	87	48.5	41.5
1.14	8–7	1.32	1.49	88	48.7	41.3
1.15	——	1.31	1.50	89	49.0	41.0
1.17	7–6	1.30	1.52	89	49.5	40.5
1.20	6–5	1.28	1.54	90	50.2	39.8
1.25	5–4	1.25	1.56	91	51.3	38.7
1.29	9–7	1.24	1.58	91	52.2	37.8
1.30	——	1.22	1.59	92	52.4	37.6
1.33	4–3	1.20	1.60	93	53.1	36.9
1.35	——	1.18	1.61	93	53.5	36.5
1.40	7–5	1.16	1.62	94	54.5	35.5
1.43	10–7	1.15	1.63	94	55.0	35.0
1.45	——	1.13	1.65	95	55.4	34.6
1.50	3–2	1.11	1.66	95	56.3	33.7
1.55	——	1.09	1.67	96	57.2	32.8
1.60	8–5	1.07	1.68	96	58.0	32.0
1.65	——	1.05	1.70	97	58.8	31.2
1.67	5–3·	1.03	1.72	98	59.1	30.9
1.70	——	1.01	1.73	99	59.5	30.5
1.75	7–4	.99	1.74	100	60.3	29.7
1.80	9–5	.97	1.75	101	61.0	29.0
1.85	——	.95	1.76	101	61.6	28.4
1.90	——	.93	1.77	101	62.2	27.8
1.95	——	.91	1.78	102	62.8	27.2
2.00	2–1	.89	1.79	102	63.5	26.5
2.10	——	.87	1.80	103	64.6	25.4
2.20	——	.84	1.81	103	65.5	24.5
2.25	9–4	.82	1.82	104	66.1	23.9
2.30	——	.80	1.83	104	66.5	23.5
2.33	7–3	.78	1.84	105	66.8	23.2
2.40	——	.76	1.85	105	67.4	22.6
2.50	5–2	.75	1.86	106	68.2	21.8
2.60	——	.73	1.86	106	68.9	21.1
2.67	8–3	.71	1.87	107	69.5	20.5
2.70	——	.69	1.87	107	69.7	20.3
2.80	——	.67	1.88	108	70.3	19.7
2.90	——	.65	1.89	108	71.0	19.0
3.00	3–1	.63	1.91	109	71.6	18.4
3.20	——	.60	1.92	109	72.7	17.3
3.33	10–3	.58	1.92	109	73.3	16.7
3.40	——	.56	1.92	110	73.6	16.4
3.50	7–2	.54	1.93	110	74.1	15.9
3.60	——	.52	1.93	110	74.5	15.5
3.80	——	.50	1.94	111	75.2	14.8
4.00	4–1	.49	1.94	111	76.0·	14.0

APPENDIX.

A FEW PAPERS ON THE

TEETH OF GEARS,

REPRINTED FROM THE

"AMERICAN MACHINIST"

AND THE

"JOURNAL OF THE FRANKLIN INSTITUTE."

These papers are not of a popular or so-called practical character, but they may be of interest to the student. There are a number of articles in the same periodicals, not reprinted here.

THE NORMAL THEORY

GEAR TOOTH CURVE.

The usual method of presenting the gear tooth curve for the examination of the student, is to devote almost exclusive attention to the minute details of the cycloidal system, to hurry over the involute system, and to get the general tooth curve, without regard to any special form, into the smallest possible compass.

The result, to the student, is a more or less intimate knowledge of the cycloid, with a fixed idea that it is the only tooth curve worth his serious attention, a smattering, generally wrong at that, with regard to the involute, and little or no acquaintance with the curve in its general and most interesting condition.

The subject should be begun at the beginning, and the beginner should learn what an "odontoid," or pure tooth curve is, what it does, and how it does it, before he is plied with epicycloids, logarithmic spirals, and the less important details of its special forms.

When properly treated, the gear tooth curve is not difficult to explain or understand, and is one of the most interesting and important applications of mathematics to practical mechanics.

Fig.1

Curve and normals
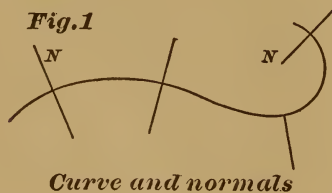
THE NORMAL.

A normal to any curve is a straight line N, Fig. 1, which is at right angles with it at the point of intersection.

Fig.2
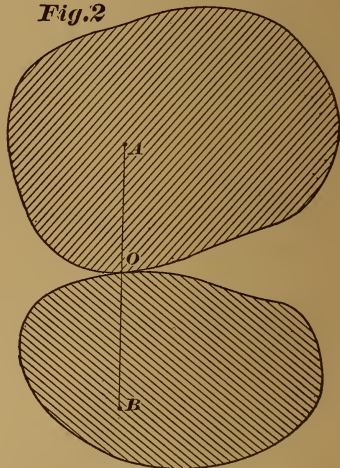
Rolling wheels

ROLLING WHEELS.

If two friction wheels, Fig. 2, roll on each other, their pitch lines touching at the pitch point O, they are, for the instant, revolving about centers A and B, which are in $A\ O\ B$, the common normal or line of centers of the two curves at their common point.

ENVELOPING TEETH.

If teeth of any arbitrary shape, Fig. 3, are fastened to one rotating wheel A, they can

be made to form teeth on a blank disk fastened to the other wheel *B*, which are the bounding curves, or "envelopes" of all their different positions.

Fig.3
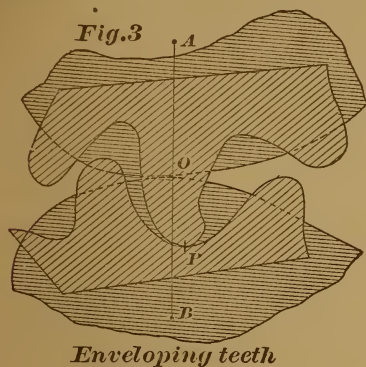
Enveloping teeth

These enveloping teeth can be formed by scribing lines about the originating teeth at short intervals of their motion, and then cutting out all the lines; or, if they are cutting tools, which reciprocate vertically, see Fig. 13, they will cut them out as the friction wheels roll together.

CONJUGATE TEETH.

If the originating teeth be passed again through the enveloping teeth they have formed, the friction wheels rolling together as at first, they will not, of course, interfere with them or cut them again. With originating teeth of certain shapes they will entirely separate at times, and touch each other at other times, while teeth of certain other shapes will have the peculiar property that they will *continually* touch each other, and not separate at all from the first touch to the last.

The latter property is evidently of the greatest mechanical value, for teeth that have it can be used to forcibly transmit a uniform motion from one rotating wheel to another, without the feeble and uncertain assistance of the friction wheels.

Such teeth are said to be "conjugate" to each other, and their curves are called "odontoids," and, as some forms are conju-

gate and others are not, the property is evidently due to some peculiarity of their shape.

Conjugate enveloping teeth will form the originating teeth by the same process, reversed, on a new wheel blank like the original, but if not conjugate they will not always reproduce the originals.

Fig.4
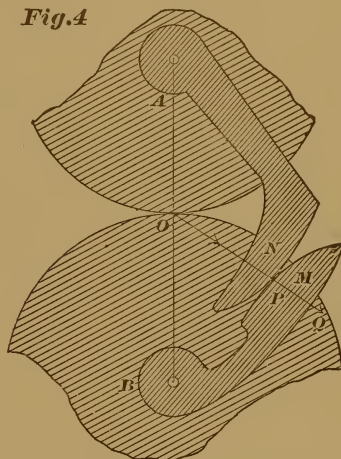
Normal intersection

THE LAW OF NORMAL INTERSECTION.

If the two odontoids, *N* and *M*, Fig. 4, are conjugate they are always in driving contact. They must always be tangent and not intersect at some one point *P*, and they must have a common normal, *O P*, at that point, which will intersect the line of centers, *A B*, at some point *O*.

The two wheels have a common velocity, *P Q*, of their common point of contact, *P*, along the common normal *P O*, and any common point *O*, on that normal, has the same common velocity in the same direction.

The two wheels can have a common velocity on the line of centers only at their common pitch point; therefore, the point *O* is that pitch point, and the first and most important law of the action of the odontoid is:

The normal to the point of contact of an odontoid with any other odontoid, always passes through the pitch point.

THE ODONTOID.

This law determines the general nature of the odontoid.

If *KP'S*, Fig. 5, is an odontoid the points *K O' G'*, etc., of the pitch line will pass the pitch point *O* consecutively, each one in its order, without a break, or a return in the order from the first point *K* to the last one used.

Fig.5
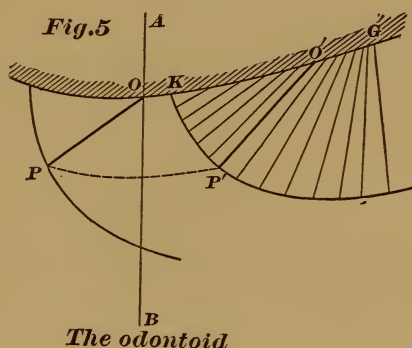
The odontoid

When any point, *P,'* of the odontoid becomes the point of contact *P*, the normal from it must pass through the pitch point *O*, and, as contact must be continuous, there must be a normal to the odontoid from every point of the pitch line.

It is also a feature of any practicable odontoid that its point of contact continually shifts on it, a new point on the curve coming into contact as each point on the pitch line comes to the pitch point, without a break or a return in the consecutive order.

When there are many normals from nearly the same point on the curve, that point is in excessive use, and such a curve, although possible, is not useful. When the normals cross each other there will be a cusp formed on the odontoid, and it is impracticable.

As both ends of each normal must come into position in order, one after the other, they must be arranged as in Fig. 5, one after the other, without a crossing, and without a blank interval. This arrangement

for want of a more expressive term, may be called "consecutive," and our general definition is :

Any curve having consecutive normals to the pitch line is a practicable odontoid.

Fig.6

Semicircular teeth
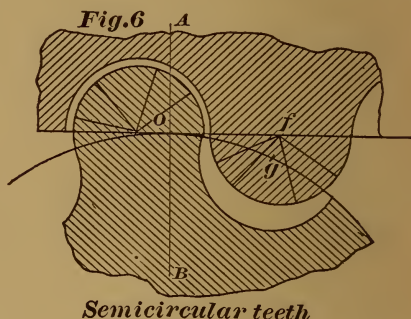
For an example of a curve that is not an odontoid, although it is often treated as such, the semicircular teeth of the rack of Fig. 6 form enveloping teeth on the pinion which are circles of the same radius. All the normals to the circle intersect the pitch line at the center *f*, and the tooth will not touch the space until the centers *f* and *g* come together at *O*. Then all the normals satisfy the law, and the tooth fits and coincides with the space.

The absence of normal intersections on any part of the pitch line shows that the teeth will separate when that part is passing the pitch point, and the junction of two or more normal intersections will show that the teeth will coincide at that place.

It has been stated that any assumed and arbitrary rack tooth, within the limitation that it is "bounded by four similar and equal lines in alternate reversion, * * * will form an interchangeable set."* The semicircular rack tooth is clearly within the given limitation, but it is not an odontoid, and will not form an interchangeable set. The conjugate tooth curve is subject to a law that is very elastic, but by no means indefinite, and which is seldom clearly given and often is given wrong by writers on this subject.

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* MacCord's Kinematics, section 283, and again at section 408. No conic section is an odontoid unless the focus is inside the pitch line, but they will all meet Prof. MacCord's requirement.
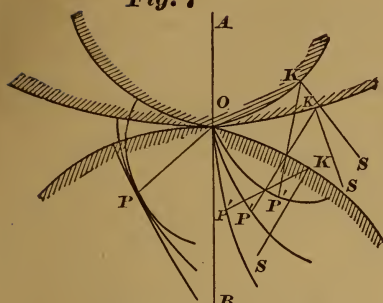
**Fig. 7**

Similar odontoids

INTERCHANGEABLE ODONTOIDS.

If any number of odontoids, Fig. 7, are formed on the same side of a set of pitch lines that will all roll together at the pitch point O, and all the odontoids conform to the requirement that the normal arcs $O\,K$, normals $P'\,K$ and normal angles $P'\,K\,S$ shall all be the same, they may be called "similar" odontoids.

As any two of these pitch lines with their odontoids roll together, the points K will pass over the same arcs and come together to the pitch point O at the same time, and as the normals and normal angles are equal, the two points P' will come together at P. The two odontoids are therefore always in driving contact, without regard to the curvature of the pitch lines, and the general law of interchangeability is:

All similar odontoids will work in interchangeable contact with each other.

Fig. 8
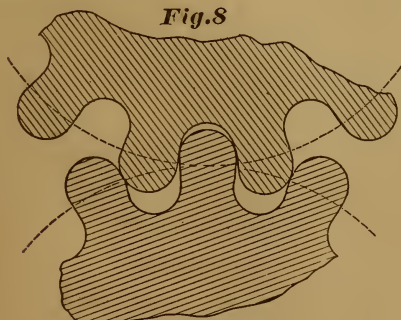
Complete teeth

FACES AND FLANKS.

The odontoid is on one side only of its pitch line and will be conjugate only to a similar odontoid on the same side of any companion pitch line, all being faces on one gear and flanks on the other.

Face gears and flank gears will work together but two face gears or two flank gears will not. To make a completely interchangeable set it is therefore necessary to provide each gear with both faces and flanks, all being similar odontoids, as in Fig. 8.

Fig. 9
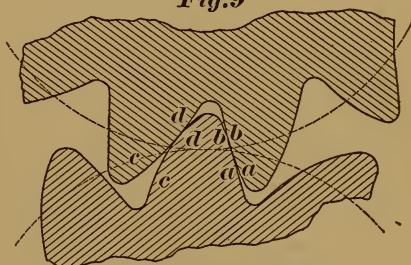
Unreversible teeth

REVERSIBLE GEARS.

If the two sides $a\,b$ and $c\,d$ of the tooth of Fig. 9 are not similar odontoids, the teeth may still belong to an interchangeable set, for the similar sides will always come together, no

Fig. 10
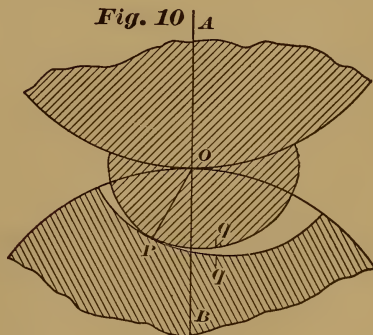
Whole teeth

matter how the gears are interchanged. But if one gear of such a set is turned over, reversed face for face, the unlike odontoids $a\,b$ and $c\,d$ will come together, and it is neces-

sary, to have the set reversible, that all four odontoids $a\ b\ c$ and d, bounding the tooth, shall be similar.

TRUNCATED TEETH.

When the point of contact P of the whole tooth, Fig. 10, is near the pitch line, its action on the mating tooth is nearly a direct push, but it becomes more and more oblique, with a wedging or crowding, as well as a pushing action, until, at the apex q there is no driving action at all, and the driven gear will stop unless both gears are so large that the next tooth is then in position, as in Fig. 8.
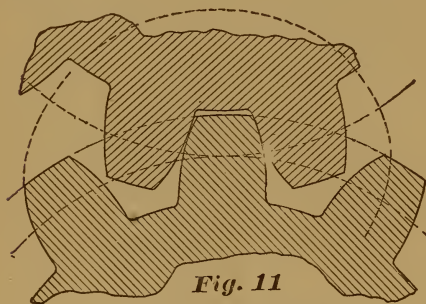
Fig. 11

Truncated teeth

To avoid this oblique action and at the same time allow the use of gears of few teeth, it is customary to truncate or cut off the apex of the curve, as shown by Fig. 11, by a line, called the addendum line, at an arbitrary distance from the pitch line.

The sides of the teeth are then brought as near together as is consistent with the required strength and we have the familiar tooth in universal use.

FORMS OF TOOTH-CURVES.

The face curve, commonly the external curve, is a lobe $O\ m$, Fig. 12, which generally returns to the pitch line, but the flank or internal curve may take a variety of shapes, generally a loop $O\ d$ or $O\ h$, but sometimes a straight diameter $O\ k$, a point at O, a loop $O f$, a cusp $O\ a\ n$, or a double cusp $O\ a'\ b\ c$.

When the flank is undercurved as at $O\ h$, a

Fig. 12

Tooth curves
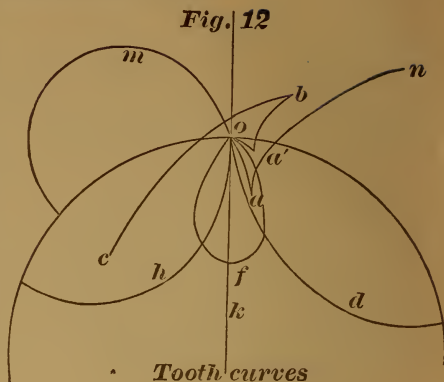
weak tooth is formed that should be avoided, and when it is nearly a point at O, it is subject to such excessive wear as to be impracticable.

When a cusp, $O\ a\ n$, or $O\ a'\ b\ c$ is formed, the action is mathematically perfect at all points, but practically is limited to the first branch from O to a. The contact changes, at the cusp, from one side of the line to the other, and is therefore impracticable with real teeth. The cusp always sets a limit to the addendum of the tooth that is working with it, for that tooth, as it continues in mathematical contact with the second branch, will interfere with and cut away the first branch.

Fig. 13
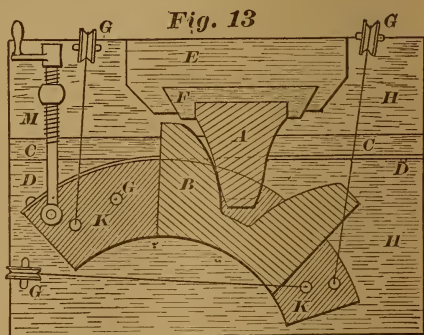
The conjugator

THE CONJUGATOR.

We have seen, Fig. 3, that any odontoid will form or "develop," an enveloping curve,

that is also an odontoid and conjugate to it. The same process provides a simple and exact method for forming templets and cutter-shapers, in the application of the theory to practical purposes.

The conjugator, Fig. 13, is here the connecting link between theory and practice, for if the gear-cutter, or templet, must be shaped by hand and eye processes, theoretical precision would be lost, and the perfection of the finished product would depend, as usual, more on personal skill than on original principles.

A rack tooth is first formed on a steel cutting tool A, which is fastened to a slide F that reciprocates vertically on a stand, E, on a plane table, H. A straight-edge, C, is fastened to the table in the position of the pitch line of the rack tooth, and an arc, K, representing the pitch line of the tooth to be formed, is rolled against it. A steel band, $D D$, keeps the arc firmly in position on the straight edge, weights $G G G$ keep it in position on the table, and a screw, M, serves to slowly roll it.

A sheet metal blank, B, for a templet, a bar of steel for a cutter-shaper, or a complete gear blank for a complete gear wheel, is fastened to the arc K.

Now give the tool A a reciprocating motion, and slowly roll the blank B past it. An odontoid will be formed on the blank that is conjugate to the rack tooth, and, if it is formed of odontoids that are similar and symmetrical with respect to the pitch line C, all the odontoids made by it will be interchangeable.

The plane table, straight-edge and arc, slide and stand, can be accurately shaped by ordinary methods, but the shaping and placing of the tool A requires considerable skill.

The chief requirement is that the rack tooth shall be formed of four equal odontoids, a, b, c, d, Fig. 14, placed symmetrically with respect to the pitch line P, and reversed with respect to the line of centers Q. If the four curves are odontoids, all the formed curves will be odontoids, but the set will not be mutually interchangeable unless they are also
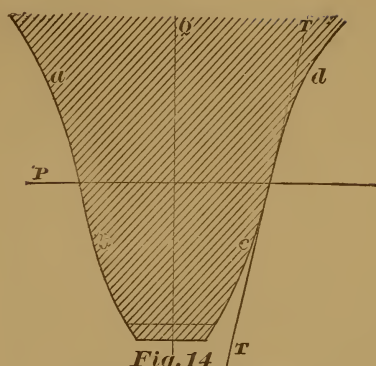

Fig. 14

Conjugating tooth

equal to each other and properly placed on the pitch and center lines. It is also desirable, although not essential, that at their junction the curves should be tangent to the same straight line T.

It is a matter of merely secondary practical importance that the originating curves should conform to some exact predetermined shape, for as long as they are odontoids the system will be perfect, if formed by the conjugator. If they are cycloids, the cycloidal system will be formed ; but if the cycloidal outline is imperfectly followed, the system may still be mechanically perfect.

The tool A should be extended beyond the addendum line, as shown by dotted lines, so that the finished gear will have the usual clearance added to the working depth of the space.

A rack tooth is chosen for the originating form on account of its simplicity. If the straight-edge C is replaced by a circular arc, the flank curves of the tool A would not be like the face curves, and an interchangeable set could be obtained only by great skill in their formation.

If the stand E tips a little out of a right angle with the table H, the cutter-shaper B will be formed with a clearance or "relief," and its deviation from a correct form will be slight. If the blank B is held on a slide that will move radially to the arc K while the cut

is being made, a relief will be given to the shaper without injury to its form.

Fig. 15

Internal teeth
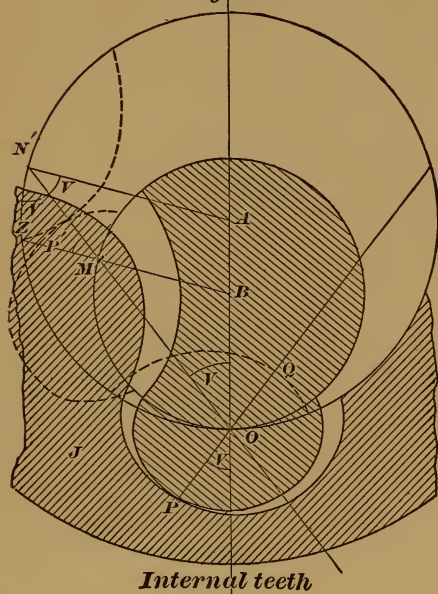
INTERNAL GEARS.

We have seen, Fig. 7, that similar odontoids are conjugate and interchangeable, whether the pitch lines curve the same or opposite ways.

When the two lines curve the same way, as in Fig. 15, the smaller gear will work inside the larger, which is then an internal gear.

The theory and its application are, in the main, the same as with external gears, the only prominent distinction being the direction of the curvature of one of the lines.

INTERFERENCE OF INTERNAL TEETH.

With internal teeth we must guard against interference, for the face of the pinion is likely to interfere with the face of the gear in a certain position P'.

When the teeth come in contact at P', their common normal N' P' M' must pass through the pitch point O. Drawing N' A and M' B,

which will be parallel, we have Z M' $N' =$ A N' M', and the normal angles are equal. As the two odontoids are similar, and the normal angles are equal, the normals P' N' and P' M' are equal, and the point of contact bisects the chord M' N'.

The normal of contact P $O = P'$ $N' = P'$ M' is therefore equal to c, the center distance A B, multiplied by the cosine of the normal angle V, or

$$\frac{P\ O}{\cos\ V} = c$$

and, if we draw the addendum at that value of P O, the teeth will clear each other, just as they would otherwise interfere. If we know the form of the odontoid in use, we can express cos V in terms of P O, and thence deduce the exact value of the latter that will let the teeth clear each other at the given center distance.

In case the face or the flank odontoids are not similar, the system is still interchangeable to the extent that any pinion will work in any gear, and in that case, the requirement to avoid interference is that the sum of the normals P' $N' + P'$ $M' = P$ $O + O$ Q must not be greater than M' N'.

LIMITING DIAMETERS.

Knowing the minimum value of $\dfrac{P\ O}{\cos\ V}$ for any odontoid that may be in use, we can determine the least center distance between two gears that will work together, for that value of $\dfrac{P\ O}{\cos\ V}$ is the required least center distance.

DOUBLE CONTACT.

If

$$\frac{P\ O}{\cos\ V} + \frac{Q\ O}{\cos\ V}$$

is always equal to $2\ c$, the two faces will always be in driving contact, and, as the face of the pinion is also always in driving contact with the flank of the gear, we have a case of double contact as far as the truncation of the tooth will permit.

If $\dfrac{O\,Q}{\cos V}$ is given, we can assume a value for $\dfrac{P\,O}{\cos V}$ that will satisfy the requirement, so that it is always possible to obtain double contact by choosing special odontoids.

Double contact is a curious but not a very valuable feature of gear teeth.

Cases of interference
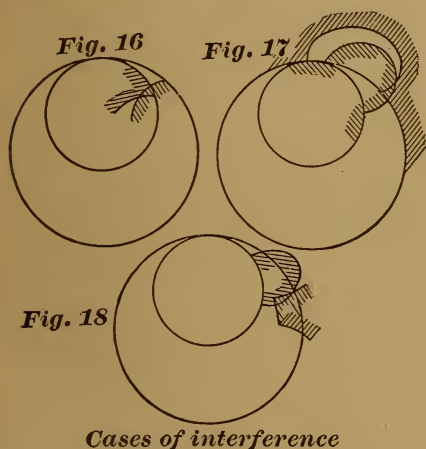
Internal gears may interfere from other causes, an odontoid from one crossing an odontoid from the other if the center distance is too small.

For example, if the tooth of the pinion of Fig. 16 is cut down to the pitch line to avoid the ordinary interference, they may still interfere as shown, the flank of the pinion interfering with the face of the gear. Similarly if the gear face is cut down, the pinion face may interfere with the gear flanks, as shown by Fig. 17.

The pinion face, when it cannot interfere with the gear face from the ordinary cause, may still cross it, as shown by Fig. 18.

The only remedy for interference is to shorten the addendum, to increase the center distance, or to use odontoids that curve quick enough to pass.

LINE OF CONTACT.

The "line of contact" is the locus of the successive positions of the point of contact of an odontoid with the mating odontoid.

As each normal, $P\,K$, Fig. 19, comes to the pitch point O, the point of contact P locates one point, P', of the line of contact, and
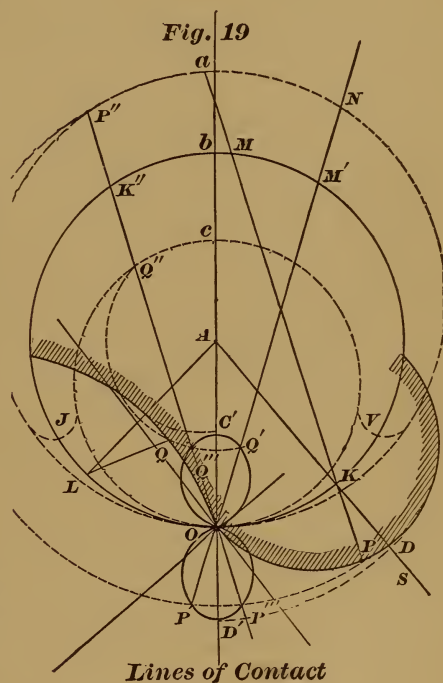
Lines of Contact

the four similar odontoids of the complete tooth form the complete line of contact $D'\,O\,C'$.

It generally takes the form of an hour-glass curve, is at right angles with the odontoid at O, and at right angles with the line of centers at D'.
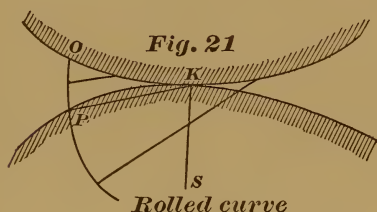
As all normals, on all similar odontoids, have the same length for the same normal angle, it follows that a system of any number of similar odontoids has a single and common line of contact, which has equal face and flank lobes.
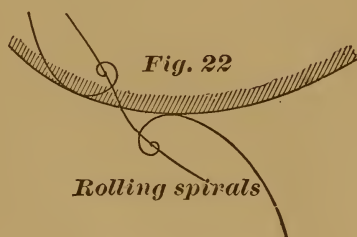
ROLLED CURVES.

If a rolling curve or "roller," $P\,K$, be rolled on a pitch line, $O\,K$, a point, P, upon it will trace out a rolled curve, $O\,P$.

As the line $P\,K$, from the tracing point P to the point of contact K, is always rotating, for the instant, about K as a center, the curve $O\,P$ will always be at right angles to it at P, and it is, therefore, always a normal to the curve. As each one of the normals is separate from the preceding and following, and the normal intersections with the pitch line are consecutive, the rolled curve is a true odontoid. As the length of the normal $P\,K$, the normal angle $P\,K\,S$, and the normal arc $P\,K = O\,K$, are in no way dependent upon

When the tracing point is any ordinary point on the roller, the curve traced will be at right angles to the pitch curve, but when it is the pole of a spiral it may cross at an angle. Fig. 22.

Although rolled curves and odontoids are identical, they cannot readily be considered the same, for the cycloid is the only odontoid worth noticing that can be conveniently handled in shape of a rolled curve.

Fig. 23

Rolled segment

Fig. 21

Rolled curve

the curvature of the pitch line $O\,K$, they are the same for the same arc $O\,K$ on all curves, and therefore all odontoids traced on the same side of any number of different pitch-curves by the same point on the same roller are similar odontoids.
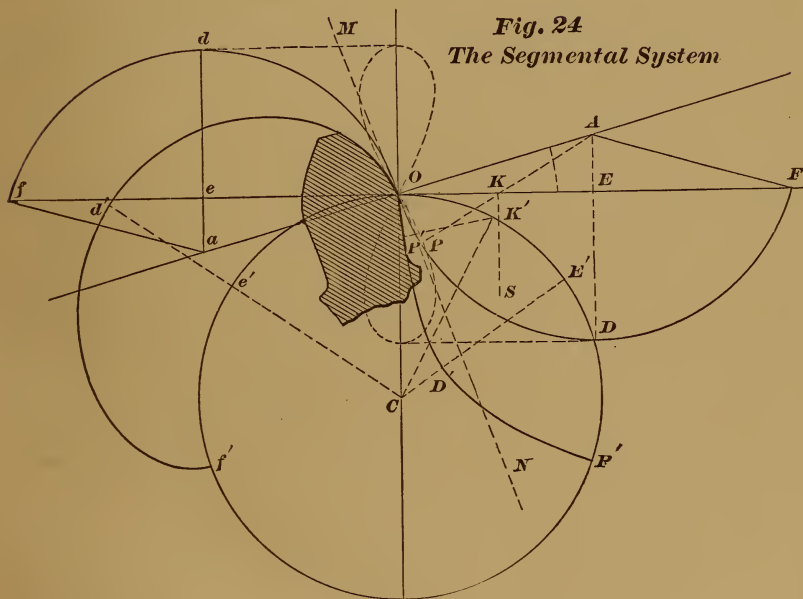

Fig. 22
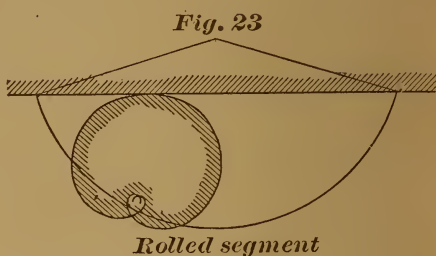
Rolling spirals

The involute can be formed by rolling a logarithmic spiral on the pitch line, but that feature is a mere curiosity without practical value, while the circular segmental tooth of Fig. 23, a perfect and very simple odontoid, can be formed only by a roller that is a curious combination of polar spirals that can be discussed only by the use of the higher mathematics.

The properties of the odontoid can generally be more easily developed and clearly explained if it is considered as a special case of the enveloping curve, than if it is treated as a rolled curve, while, for practical purposes, the conjugator, founded on the normal theory, has the advantage of any device that is founded on the rolled curve theory.

APPLICATION OF THE THEORY

TO

PARTICULAR CASES.

Fig. 24
The Segmental System

THE SEGMENTAL SYSTEM.—If a circular arc be drawn from a center A, Fig. 24, on a line at an angle, $E\ O\ A$, with a rack pitch line, $O\ F$, its normals $P\ K$ to the pitch line, will satisfy the law of the odontoid, and the segment $O\ D\ F$ will be a rack tooth that will form an interchangeable or conjugate set of teeth.

If the radius $O\ A$ is infinite, the segment is a straight line $M\ O\ N$, at right angles at O with $O\ A$, and the common involute system will be formed. Therefore the involute tooth is a special form, the infinite form, of the segmental tooth.

The segment has exactly the valuable properties of the involute at the pitch line, and approximately away from it, the approximation being closer as the radius $O\ A$ is longer.

The involute tooth is often, but not properly, regarded as the special case of the cycloidal tooth for a rolling circle of infinite diameter. Regarded simply as a curve, the involute is an infinite cycloid, but regarded as a gear tooth curve it is not, for, as shown by Fig. 37, infinite cycloids have a mathematical but not a practicable contact, and cannot bear properly, unless the conditions of the movement are so far strained that one is

reversed on its pitch line, and then the pitch lines are separated.

FORM OF THE SEGMENTAL TOOTH.—The face of any segmental tooth, on any circle with center at C, will be a lobe, $O\,d'\,f''$, where $e'\,d' = e\,d$, and $O\,e'\,f'' = O\,ef$. It is always at right angles at O with $O\,a$.

The flank curve $O\,D'\,F''$ is at right angles at O with $O\,A$, has the same height $D'\,E''$, and the same base $O\,E'\,F''$ at the rack face.

If a, O and A are in the same line, the face and flank will join at O, and be a single curve.

no cusp will be apparent, but the slightest increase of the proportion will separate the points.

In the most convenient system, where $AOE = 14°\ 28'\ 40''$, and sin. $AOE = \frac{1}{4}$, the cusps will appear whenever the segment and pinion radii are in the proportion

$$\frac{OA}{OC} = \tfrac{27}{4} \cdot \tfrac{1}{4} = 1.687.$$

As the proportion $\dfrac{OA}{OC}$ increases, the second branch $Q'\,R$ will increase so that the curve will take the form $O\,Q'\,R'\,D''$, and when OA
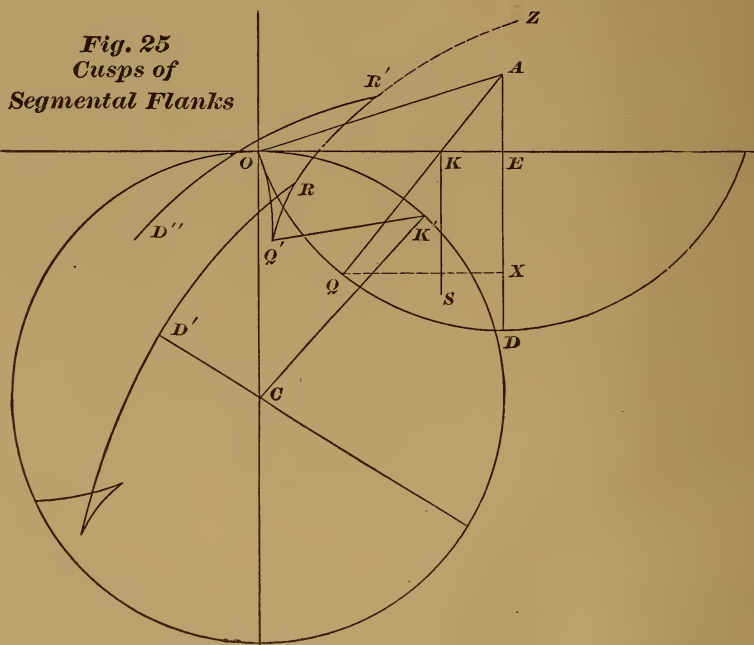
Fig. 25
Cusps of
Segmental Flanks

CUSPS OF SEGMENTAL FLANK.—When the radius $O\,A$, Fig. 25, is small, compared with the radius OC, the pinion flank takes the form shown by Fig. 24; but as the proportion $\dfrac{OA}{OC}$ increases, a value will be reached, when $\dfrac{OA}{OC} = \tfrac{27}{4}$ sin. AOE, at which a double cusp, $Q'\,R$, will form. At exactly that point the two points Q' and R will coincide, and

is infinite the second branch, $Q'\,Z$, then an involute, is infinite.

THE SEGMENTAL DELINEATOR.—The segmental curve can be formed by the "conjugator" previously described, and shown by Fig. 13, and it can be drawn by the special delineator, shown by Fig 26.

A thin wooden wheel, C, turns on a pin at its center, and a rack, B, rolls on it, being held to it by a strip, aOc, of thin brass or

strong paper attached to both. It is kept in position by a guide, H.

A fixed bar G, projecting over the wheel, carries a pin O, placed exactly at the point of contact of wheel and rack. A rule E turns about a pin A in the rack B, and carries a pointed tracing pin or pencil point at P. The pins A O and P are in line, and all three are always at the same distance from the straight edge of the rule E. The pin P will pass under and come in line with the pin O.

As the rack is rolled on the wheel, the rule will turn about the pin A and slide on the pin O. The point P will trace segment of

The action is practicable until the point of contact arrives at the first cusp Q' of Fig. 27; but beyond that, when it is on the second branch Q' R, the flank curve is inside the rack face, and the action is impracticable.

There will also be an actual interference with the first branch. When the point of contact is on the second branch, the rack face will cross the first branch at J, and therefore the addendum must terminate the rack tooth at the point Q that conjugates with the cusp Q'.

The difference between theoretical and practical contact is illustrated by the two ma-

Fig. 26

Fig. 26
Segmental Delineator
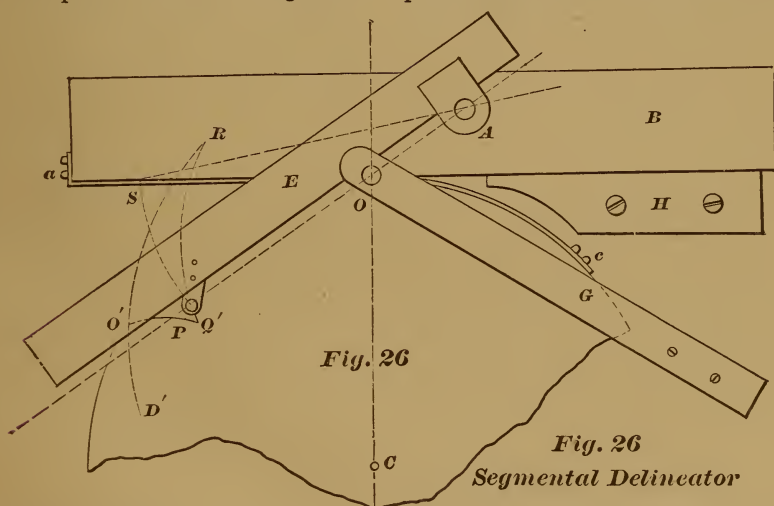
a circle P S with respect to the rack, but on the wheel will trace out the segmental flank O' Q R D'.

If the pin A is carried by an arm on the other side of the rack pitch line, the face of the pinion tooth will be drawn, but, as the form of the face is very simple, the utility of the instrument is confined to the flank curve.

INTERFERENCE OF SEGMENTAL TEETH.—The action of the segmental rack tooth on a flank that is conjugate to it, when the proportion is such that a cusp is formed, is always mathematically perfect, but not always practicable or capable of mechanical use.

chines, the conjugator of Fig. 13 and the delineator of Fig. 26. A full rack tooth on the conjugator will form the first branch correctly, but when the cusp is reached will return on it and cut it away, while the delineator, having but one acting point, will follow the theory and trace out all three branches.

LEAST NUMBER OF TEETH.—Therefore, if the addendum is fixed, and it usually is, interference will generally set a limit to the diameter of the smallest pinion with which a rack tooth having the given addendum will work, without bearing on the second branch of the pinion flank.

The diametral pitch being unity, a the ad-

dendum EX, Fig. 25, b, the segment radius OA, and V, the angle of obliquity AOE, the smallest possible number of teeth is

$$t = \frac{2a \sin V}{\left(\frac{a}{b} + \sin V\right)^3} \qquad (1)$$

If $b = \infty$ for the common involute system, $a = 1$, and sin. $V = .25$, this formula gives $t =$

Interference of Segmental Teeth
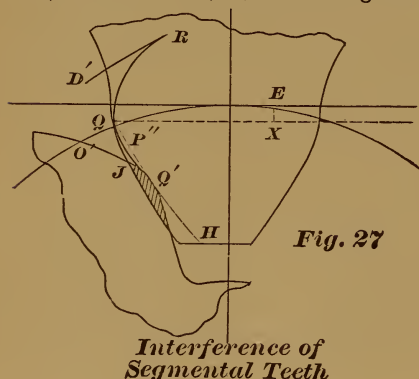
32. Therefore, the common involute system cannot have an addendum of unity on an interchangeable set having gears with less than thirty-two teeth. When the set includes 12 teeth, as is usual, the addendum must be shortened, or the points must be rounded over, as at $QQ'H$, Fig. 27.

If we have given the angle of obliquity, the addendum, and the number of teeth in the smallest pinion, the largest possible segment radius that can be used is

$$b = \frac{a}{\sqrt[3]{\dfrac{2a \sin. V}{t}} - \sin. V} \qquad (2)$$

This, for the common case, where $a = 1$, $t = 12$, and sin. $V = .25$, gives $b = 10.34$ as the radius for the usual twelve-tooth system. A radius of 13.91 will allow 15 teeth, 16.95 will allow 16.95 teeth, 23.58 will allow 20 teeth, and a short radius of 8.44 will admit a 10-tooth pinion.

THE NATURAL SET.—There is one particular proportion of segment radius to pinion radius, that might be considered the natural limit to the interchangeable system, and that

is the proportion at which the cusp first appears. If that, or a smaller proportion is chosen, there is no limit set to the addendum, and no interference is possible, for the second branch of the curve never appears.

For that point we have the relation

$$b = 3\tfrac{3}{8} \, t \sin. V, \qquad (3)$$

so that, by choosing some value of t as the lower limit, we can find b for the whole set. If sin. $V = .25$, we find $b = \tfrac{27}{32}t$, giving $b = 8\tfrac{7}{16}$ for a ten-tooth set, $b = 10\tfrac{1}{8}$ for a twelve-tooth set, $b = 12\tfrac{21}{32}$ for a fifteen-tooth set, $b = 27$ for a thirty-two-tooth set, and so on.

If we use the plan previously explained, and calculate by formula (2), we can get a greater value for it, but in that case the addendum is limited to its chosen value.

As the addendum is always limited in practice, almost always being unity, formula (2) appears to be better adapted to practical purposes than formula (3).

CORRECTED INVOLUTE TOOTH.—We have seen that the true involute tooth, when sin. $V = .25$, cannot be used for an interchangeable set

Stunted Involute
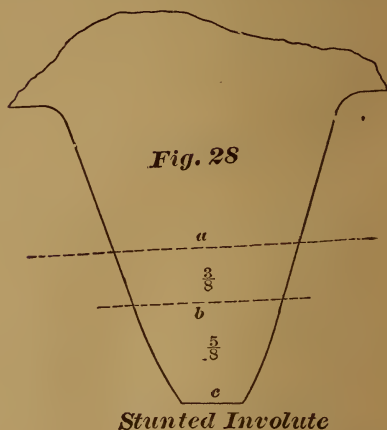
that includes gears with less than thirty-two teeth, if the addendum is unity.

It is, however, customary to use the full addendum on a set that includes twelve teeth with the result that it must be corrected (?) for interference by rounding over the corners as in Fig. 27.

Formula (1) will apply to the involute if $b=\infty$. In that case $\frac{a}{b}=o$, and

$$a=\frac{t \sin.^2 V}{2} \qquad (4)$$

If $t=12$, and sin. $V=.25$, we have $a=\frac{3}{8}$, so that the common involute, Fig. 28, is limited to the addendum $ab=\frac{3}{8}$, and the additional $bc=\frac{5}{8}$ of the full addendum must be cut off or got out of the way by rounding off as much of it as would interfere with a twelve-tooth pinion.

This additional five-eighths is not interchangeable, is not a tooth curve, and is kept on merely to give the appearance of a whole tooth. What usually appears to be a full addendum, is really stunted to but little more than its third part.

Fig. 29

pitch line

Action of
Corrected Involute Tooth
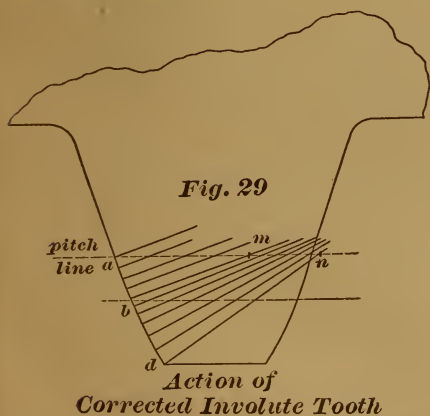
The only true correction, the only device that will allow of a full addendum of unity, retain the true involute for any part of it, and permit a rack to run in a pinion of less than thirty-two teeth, would be to correct the rack tooth by rounding over the point as in Fig. 29, to give the flank the same correction, so that the condition of interchangeability is satisfied, and then to form a conjugate set from the corrected rack tooth. The result would be a mixed action : true involute, near the pitch line, and epicycloidal or otherwise away from it.

This plan would have the serious defect that the corrected part bd must be a very defective odontoid, with a jerky and very nearly impracticable action ; for, to obtain the necessary correction between b and d the curve must turn so quickly that its normal intersections with the pitch line must be crowded within a narrow limit mn.

Therefore, it would not appear to be advisable to correct the involute at all, for low-numbered pinions, but to discard it altogether, or to keep up appearances, as at present, by a merely ornamental and deceptive extension to nearly three times its effective length.

If it is discarded, its valuable peculiarities will be lost, and therefore its substitute should be the curve that is nearest like it, and most nearly has its properties.

Evidently, the nearest possible approach to the involute, is the segment that has the same angle of obliquity, and the longest radius that will admit the required addendum on the required smallest pinion, as found by formula (2).

Figs. 29 and 30 serve to compare the action of the corrected involute with the segmental tooth. The action of the segment, Fig. 30, is exactly the same as that of the involute at

Fig. 30

pitch line

Action of
Segmental Tooth
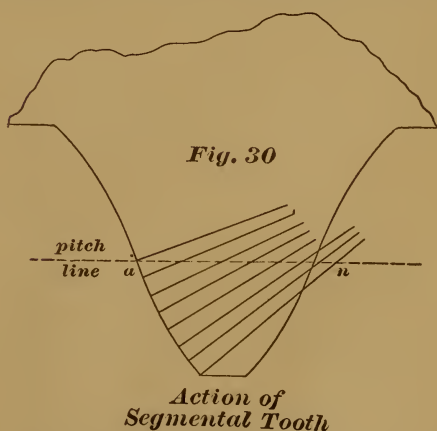
a, and its rapidity gradually increases to the finish at n. The corrected involute action, Fig. 29, is uniform from a to m, and finishes with a sudden jerk from m to n.

If the correction should be curved enough to cause any of the normals to meet on or outside of the pitch line, the action would be wholly impracticable.

THE CYCLOIDAL SYSTEM.—The cycloidal system is generally, but not properly, called the "epicycloidal" system. It is no more epicycloidal than it is hypocycloidal, for the faces are of the one form, and the flanks of the other. It is simpler and easier, as well as more correct, to apply the name cycloidal to both face and flank, and to the whole system, as is sometimes done.

As before stated, the cycloidal system can be more easily developed and studied by the "rolled curve" theory than by the normal
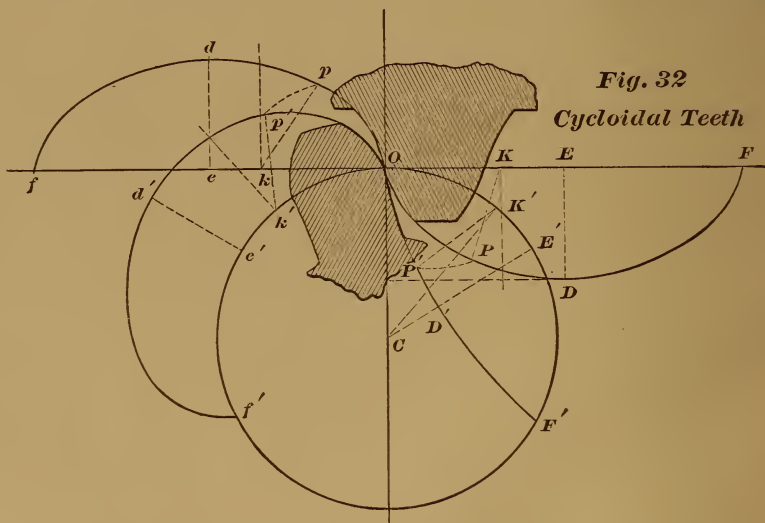
angles at O and F, with a base, OF, equal in length to the circumference of the roller.

As with all rolled curves, the line PK, from the tracing point to the tangent point, is al-

Fig. 31
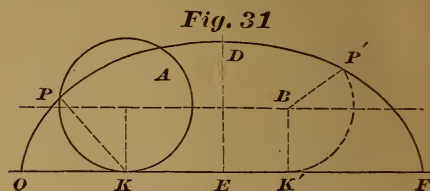
The Cycloid

ways a normal to the curve, and therefore, to draw a normal to any given point P, strike a circle through the point having a diameter

Fig. 32

Cycloidal Teeth

theory, because its roller, the circle, is the simplest of all curves. But, in this place, the former theory will not be used further than to define the nature of the cycloid, which is the generating odontoid that forms the system.

THE CYCLOID.—If a circle A, Fig. 31, is rolled on the straight line OF, a fixed point P, in it will trace out a transcendental curve, $OPDF$, called the cycloid.

It is a lobe, having a height, DE, equal to the diameter of the roller, and is at right

equal to the height DE, and tangent to the base line, and draw the normal PK to the point of tangency.

As the arrangement of the normals is consecutive, the curve is an odontoid, and all curves formed from it will be similar odontoids that will work interchangeably with it.

The simple process for drawing the normal makes it easy to form the conjugate face or flank belonging to any pitch circle. The flank cycloid Odf, Fig. 32, forms a face on the pinion, which is always a lobe $Od'f'$ at

right angles at O with the pitch line, and meeting it again at f'', where $Oe'f'=Oef$.

The face cycloid ODF forms a flank, $OD'F'$, on the pinion, that is at right angles with the pitch line at O, meets it again at F', where $OD'F'=ODF$, and which takes various forms, according to the size of the pinion compared with that of the cycloid.

When the radius OC, of the pinion, is greater than the height ED of the cycloid, the flank will be a concave lobe, $OD'F'$.

Fig. 33

Radial Flanks
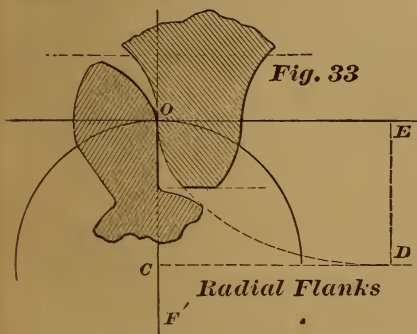
When the radius OC is equal to the height ED, as in Fig. 33, the flank will be a straight diameter OF'.


Fig. 34

Reversed Flanks
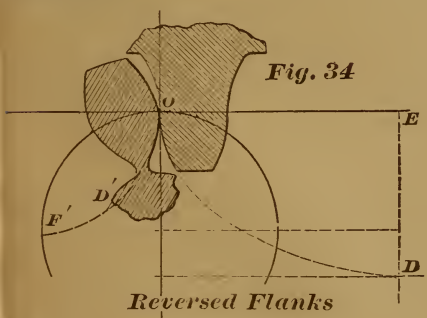
When the radius is less than the height, as in Fig. 34, the flank will be a convex lobe.

As an undercurved flank, as in Fig. 34, is weak, it is customary to so limit the radius of the pinion that it shall never be less than the height of the originating cycloid.

As the proportion of ED to OC still further increases, the flank is still more undercurved, until when $OC=\frac{1}{2}ED$, we have the base, OF,

equal to the circumference of the pinion; and the flank is concentrated to a single point at O. The wearing action is also concentrated at the single point, and such a tooth, although practicable, is quite useless.

Fig. 35
Inverted Flanks
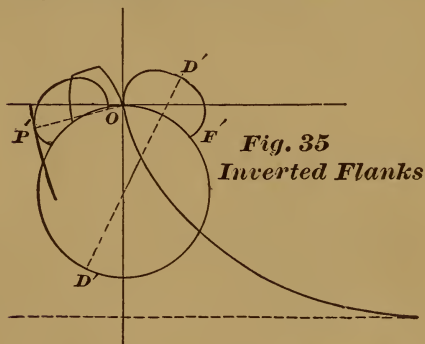
If the height of the cycloid is greater than the diameter of the pinion, Fig. 35, the flank is a lobe, entirely external to the pitch line; and although the contact is still mathematically perfect, it is no longer practicable, for it is on the inside of the cycloid, as shown at P'.

Fig. 36

Cycloidal Involute
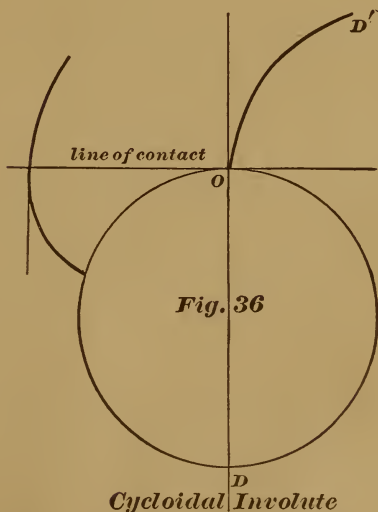
If the proportion is carried to its extreme, the height being infinite as compared with the diameter of the pinion, as in Fig. 36, the

cycloid becomes the straight line OD, and the pinion flank is the involute OD'. From this it is plain that the involute, in the form of an infinite cycloidal odontoid, is not a practicable gear tooth curve, the action between two gears being, as in Fig. 37, always on the straight line AOB at the crossing of the two tooth curves.
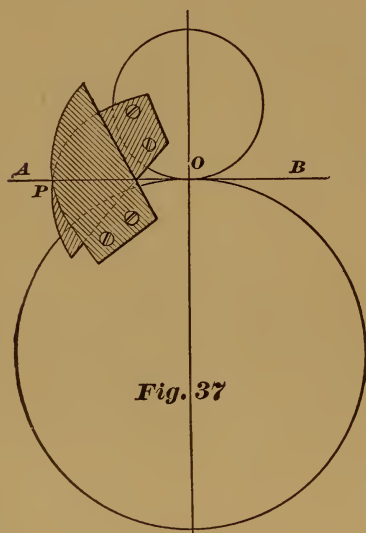

Fig. 37

Infinite Cycloidal Teeth

CYCLOIDAL LINES OF CONTACT.—The primary lines of contact, in the case of the cycloid, are circles, Fig. 38, of the same diameters as the heights of the originating cycloids.

The secondary lines of contact are also circles. The diameter is equal to the pitch diameter plus or minus the height of the cycloid.

INTERNAL INTERFERENCE.—With cycloidal teeth we cannot have a partial interference, as with segmental teeth, which can be remedied by truncation of the teeth, for the exterior secondary of the pinion is either entirely inside the interior secondary of the gear or entirely outside of it, except in the case of entire coincidence.   If the differ-

ence between the pitch diameters is greater than the sum of the heights of the originating cycloids, there can be no interference, but if it is less there will be a continual interference that can be remedied only by the entire removal of the face of one of the teeth. The condition of non-interference can be conveniently expressed by the rule that the difference between the numbers of teeth on the gears must not be less than the half sum of the numbers of teeth on the base gears. This, for the common interchangeable system, requires that there should be a difference between the gears at least as large as the base gear.   For example, in the fifteen tooth set there must be at least fifteen more teeth in the gear than in the pinion.*

DOUBLE INTERNAL CONTACT.—When the condition of non-interference is exactly satisfied, there is a case of double contact, for then the two secondaries coincide.   In a case of double contact of interchangeable teeth, the coinciding secondaries must exactly bisect the chord $c\,d$ of Fig. 20, for the primaries are equal, and their chord $e\,f$ is exactly bisected in that case.   As the circle is the only curve that will bisect all the chords $c\,d$, it follows that the cycloidal system is the only one that can have double contact, and at the same time be interchangeable.

PRACTICAL CONSTRUCTION.—The practical application of the conjugating process, in the case of the cycloid, presents the difficulty that the originating rack tooth must be an exact cycloid, which condition can be met only by special mechanism.

The originating segmental tooth has an outline that is formed of arcs of circles, and that of the involute is composed of straight lines, and both can be easily shaped.   But the difficulty in the practical application of the cycloidal system is not by any means the greatest objection to it in comparison with its simpler and superior rival.

* The discovery of the law of internal interference, as far as it relates to cycloidal teeth, is generally credited to Prof. C. W. MacCord ; but, in claiming that discovery, the professor could not have been aware of its previous publication, by A. K. Mansfield, in the Journal of the Franklin Institute for January, 1877.
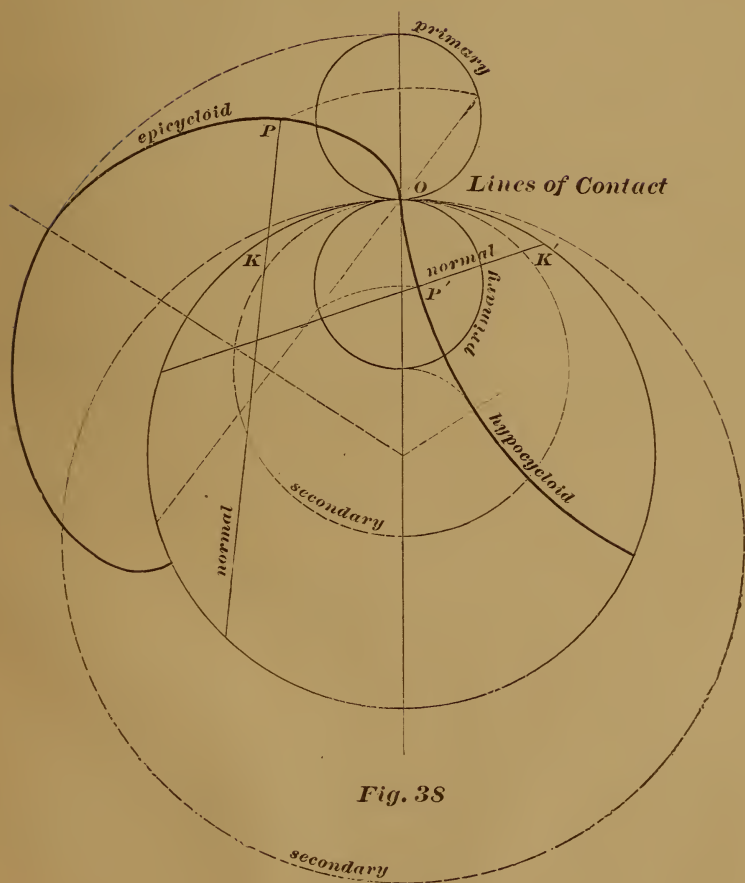
Fig. 38

The COMPARATIVE EFFICIENCY of the TEETH of GEARS.

The effect of friction between the teeth of gears is not well understood, and the popular impression, even among educated engineers, concerning the comparative efficiency of the two forms of teeth in common use—the involute and the cycloidal—is that the latter is much the most economical, and, therefore, much better adapted for use for the transmission of heavy power.

This impression is entirely wrong, the reverse of the provable facts, and it is based not entirely on fancy but partly on the teaching of authorities that are undoubtedly competent.

It is with no small feeling of timidity, that I venture to contradict the declared and apparently proved opinions of such high authorities as Reuleaux, Herrmann and others, and I would not dare to assert a contrary view if I did not feel able to prove it, by evidence that will bear the closest examination. I will give the demonstration in great detail, so that it can be followed by any one who is familiar with the common processes of analysis.

By the work done by a gear wheel, I mean the work done by the friction of sliding between the teeth. I shall leave out the small rolling friction between the teeth, and I shall not consider the friction of the shaft bearings.

The work lost by the rubbing of two surfaces on each other is the product of the normal force acting between the two surfaces, by the distance through which the resistance is overcome, and by the coefficient of friction for the material in use.

To determine the work done by a pair of gear teeth, we must determine these three factors or their product, and this may be done in two different ways: by a graphical process, and by an analytical method. The two processes are entirely independent of each other beyond the given premises, and their agreement upon a common result is a substantial proof of the accuracy of both.

Graphical Process.—In *Fig. 1*, the two tooth curves have rubbed upon each other, while the point of contact between them has moved from C to A on the line of action, $O A D$, and they have

done work that is the product of the coefficient of friction, f, by the difference, zy, of the lengths of the curves that have passed the point of contact, and, for graphical purposes, of the average force, S', that has acted between the two teeth.

If we make a drawing, showing the two teeth in several positions, preferably at equal intervals of their action, we can determine the work done within the limits of each interval by multiplying together the factors as found by measurement. The total work done between any two points is the sum of these products for all the intervals between the points.

In *Fig. 2*, this process is applied to an exaggerated example of a pair of cycloidal teeth. The gears, with radii k and h, have ten
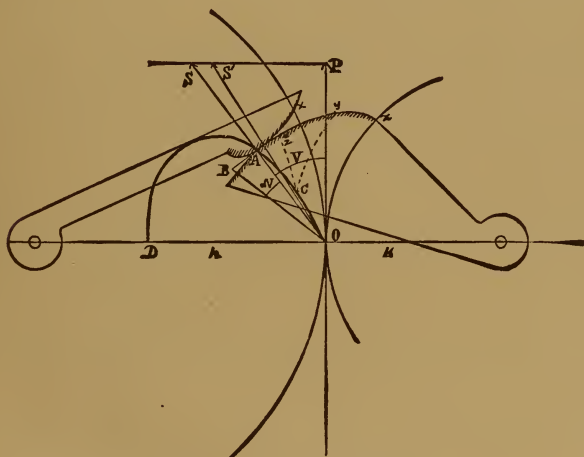

FIG. 1. Analytical Process.

and twenty teeth, the tangential force, OP, between the two gears is assumed to be constant and unity, and the coefficient of friction is assumed to be one-tenth. The describing circle, with radius $OM = r$, has three teeth, so that a gear of six teeth would have radial flanks and be the base or smallest gear of the interchangeable set to which the two gears belong.

The pitch and describing circles are divided into equal intervals, Oa, ab, bc, etc., of one-twelfth of the whole tooth arc, or circular pitch, Ol, commencing at the line of centres, and the work done over each of these small intervals is to be determined.

Make a templet of an epicycloid on the gear h, and of a hypocycloid within the gear k, and draw curves from each of the divisions of the pitch lines. Each pair of curves should meet on the

corresponding division of the describing circle. Measure the differences between the lengths of these curves (see column 2 of the table), and by subtracting each total difference from the next larger, find the partial length of curve passed over during each interval, as tabulated at column 3.

Draw a line at an arbitrary distance, representing unity, from the line of centres and parallel with it, and draw lines, *OSa, OSb, OSc,* etc., through the centres of the intervals. The length of each line (column 4) can, with small error, be assumed to be the average normal force for its interval. These normal forces can be very
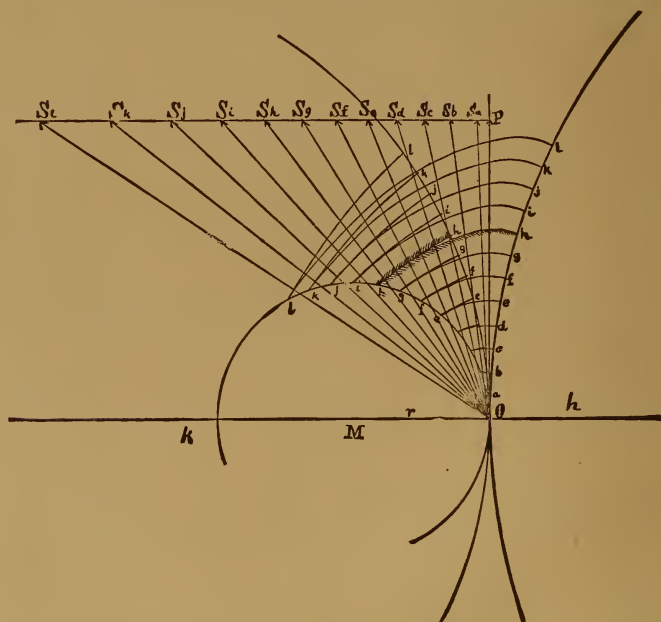

FIG. 2. Graphical Process.

easily computed, for each one is the reciprocal of the cosine of the angle POS. The angle for the first normal is $2\frac{1}{2}°$, and there are $5°$ between each of the following normals:

Multiplying together the normal for each interval, the partial curve for that interval, and the coefficient of friction, we obtain the loss for each interval as tabulated at column 5. By summation we obtain the total loss to and including each interval, as tabulated at column 6.

For the involute tooth, we have a constant normal force, $S = 1\cdot15$, the total work done, column 10, up to any interval is the product

of that force by the total curve, column 9, for that interval. The figure is so similar to *Fig. 2*, that it need not be given here.

The graphical process will determine the general result, and show that while the two curves are substantially equal in efficiency, the advantage is a very little in favor of the involute. If we wish a precise comparison between these two curves, no graphical process can be used, and we must resort to analysis.

Analytical Process.—In *Fig. 1*, the two tooth curves are odontoids of any possible form, and they will secure a uniform velocity ratio between the pitch lines. They slide on each other, the point of contact moving along the line of action, $O\,A\,D$. At any time they are at a distance $O\,A = b$ from the pitch point, O, and are pressed together with a normal force, S, which is equal to the constant tangential force, P, divided by the cosine of the angle of obliquity, $P\,O\,A = V$, and this normal force is always in the direction of the pitch point O.

While the normal, $O\,A$, turns through an elementary angle, the arc of which is $d\,V$, the two curves will rub on each other over an elementary distance, $A\,B = O\,A \cdot d\,V = b \cdot d\,V$, and they will do the elementary work

$$d\,W = f\,P \cdot \overline{A\,B} \cdot S \cdot = f\,\frac{P}{\cos\,V} \cdot b \cdot d\,V.$$

At the same time the wheel h will turn through the elementary angle, the arc of which is

$$d\,x = \frac{k}{k \pm h}\,d\,V$$

in which the positive sign is for external, and the negative sign is for internal contact.

Therefore, we have the total work done by friction, while the wheel h is turning through an angle, the arc of which is x.

$$W = f\,P \cdot \frac{k \pm h}{k} \int_{0}^{x} \frac{b\,d\,x}{\cos\,V}$$

and this cannot be carried further until we know the form of tooth curve to be used, and can determine b and $\cos\,V$ in terms of x.

First take the involute tooth.

The distance $O A = b$ is equal to $h x . \cos V$, and we have the total work done

$$I = f P . \frac{k \pm h}{k} h \int_0^x x\, d x,$$

which integrates to

$$I = \frac{f P}{2} . \frac{k \pm h}{k} h \frac{x^2}{2},$$

or, if we use the arc on the pitch line, $w = h x$, we have

$$I = \frac{f P}{2} . \frac{k \pm h}{k h} w^2$$

for the value of the work done by the friction of involute teeth while moving from the pitch point over any arc, w, on the pitch circle.

It is a singular fact that this loss of power is the same for all values of the angle of obliquity. All involute systems are equal in efficiency, without regard to the angle of obliquity.

Then take the cycloidal tooth.

We have $b = 2 r . \sin \frac{h}{2 r} x$, and $\cos V = \cos \frac{h}{2 r} x$, giving the total work.

$$E = f P . \frac{k \pm h}{k h} 2 r \int_0^x \tan \frac{h x}{2 r} d x,$$

which integrates to

$$E = - f P . \frac{k \pm h}{k h} 4 r^2 \ nat \ log \ cos \ \frac{w}{2 r},$$

the value of the total work of a pair of cycloidal teeth.

————

To compare the cycloidal with the involute tooth for the same arc of action from the pitch point, divide E by I.

$$\frac{E}{I} = \frac{8 r^2 \ nat \ log \ cos \ \frac{w}{2 r}}{w^2}$$

As this is unity for $w = 0$ and greater than unity for any finite value of w, it follows that the efficiency of the involute is mathematically superior to that of the cycloidal curve, in all cases and under all circumstances, without regard either to the angle of obliquity of the involute, the size of the describing circle of the cycloidal curve, or the arc of action, and provided only that the

comparison is made over the same arc of action. (See column 13 of the table.)

In both of these formulæ it is seen that h and k can exchange places without affecting the result for external contact, and therefore the work done is the same, for the same arc of action, on both sides of the line of centres, the tangential force being constant.

For a comparison between external and internal gears, we have

$$\frac{A}{B} = \frac{I \text{ or } E \text{ Ext.}}{I \text{ or } E \text{ Int.}} = \frac{k + h}{k - h}$$

so that the internal gear is much the most economical, particularly when the two gears are nearly of the same size.

When $k = 2h$ we have $\dfrac{A}{B} = 3$. That is, if the internal gear is twice the size of its pinion, the work lost is but one-third of that lost when both gears are external.

Small improvement can be made by putting a small pinion inside, rather than outside of a large gear, as is often done at great expense on boring mills and large face plate lathes. A six-inch pinion and a six-foot gear will give $\dfrac{A}{B} = 1 \cdot 18$ an advantage of no great value.

It is seen from the above that the work being done increases very rapidly with the arc of action; with the square of that arc in the case of the involute, and in a still greater proportion for cycloidal teeth, and hence that arc should always be made as small as possible.

Strength should be secured by a wide face rather than by a large tooth, for the face of the gear has no influence on its efficiency.

The two formulæ for E and I can be very easily applied to any particular example, and the results obtained much more quickly, as well as more accurately than by the graphical method.

For application to the given example, where $h = 10$, $k = 5$, $f = 1$, and $P = 1$, we have

$$E = 6 \cdot 2170 \,[\,C - log \, cos \, (5 \, n)°\,]$$
$$I = \cdot 01028 \, n^2$$

in which n is the number of any interval, C, is the characteristic

with the sign changed, and *log cos* contains only the mantissa of the common logarithmic cosine of $5 n^o$.

It is seen from the tabulated value of E and I obtained by computation, columns 7 and 11, that the graphical and analytical processes agree very closely, the errors being shown by columns 8 and 12. As before stated, this agreement is a strong indication of the accuracy of both.

Prof. Reuleaux* finds that the two curves are exactly equal when compared over the same arc of action, and Prof. Hermann† finds the same result by a different process. In both cases the result was arrived at by making an approximation, for reasons not given but probably to simplify the work.

If the actual determination of the work done is the end in view, the approximations can be allowed, as the result is then close enough for all practical purposes. But, if the object is a close comparison between the two curves, the slightest difference must be accounted for, and neither Reuleaux's nor Herrmann's formulæ will answer the purpose.

Herrmann remarks, "It is evident, moreover, that the friction of involute teeth will be somewhat greater than that of cycloidal teeth, the angle γ being smaller for the former than for the latter."

This may be "evident," but it is not provable, and the statement that the angle γ, which is the complement of the angle of obliquity, is smaller for the involute, is not correct. Up to the half tooth point it is so, but beyond that point the reverse is true. At the half tooth point the two forms always have the same angle of obliquity if they belong to interchangeable sets which have the same base gear.

Further, it does not follow that the work of friction is the greater when the angle of obliquity is the greater, for the work of friction depends on two variable factors, the normal pressure, which indeed increases with that angle, and the length of the curve that is rubbed over. Within the half tooth point this curve is the shortest for the involute, so that the work done is the smallest although the other factor is the greatest.

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* *Transactions of the American Society of Mechanical Engineers*, vol. viii, 1886. The result, without the demonstration, is also given in *Reuleaux's Konstrukteur*, § 213.

† Klein's translation of Herrmann's revision of *Weisbach's Mechanics of Engineering and Machinery*, vol. iii, § 79.

As Herrmann states, "This difference is insignificant for the tooth profiles ordinarily employed," but the general impression, which it is the object of this paper to contradict, is that the difference is very significant and in favor of the cycloidal tooth.

Reuleaux goes further, and, after finding that the two curves are exactly the same for the same arc of action, gives several practical examples, which show the involute to be decidedly inferior, the difference being from sixty to eighty per cent.

This result is correct for the conditions of Reuleaux's examples, but it seems to me that those conditions are not correct if the object is to compare the two curves, for he does not take them on the same terms. He takes the involute with a long arc of action, and compares it with a cycloidal tooth having a short arc, and of course the involute is then inferior.

EXAMPLE FOR $h = 10°$ $k = 5°$ $r = 1·5$ $f = ·1$ AND $P = 1$.

INTERVAL.	CYCLOIDAL TEETH.							INVOLUTE TEETH. OBLIQUITY, 30°. S = 1·15.				$\dfrac{E}{I}$
	Total Curve	Partial Curve	Normal Force.	Partial Work	Total Work.			Total Curve	Total Work.			
					Graph	Anal's	Error.		Graph	Anal's	Error.	
1	·010	·010	1·0009	·0010	·0011	·00103	·0001	·02	·0023	·00103	·0013	1·002
2	·035	·025	1·0087	·0025	·0036	·00413	·0005	·04	·0046	·00411	·0005	1·004
3	·085	·050	1·0243	·0510	·0086	·00937	·0008	·08	·0092	·00924	0	1·013
4	·155	·070	1·0485	·0735	·0160	·01679	·0008	·15	·0173	·01645	·0008	1·021
5	·245	·090	1·0824	·0975	·0257	·02656	·0009	·22	0254	·02570	·0003	1·033
6	·355	·110	1·1274	·1240	·0382	·03884	·0006	·32	·0370	·03701	0	1·049
7	·485	·130	1.1857	·1540	·0536	·05386	·0003	·44	·0508	·05038	·0004	1·069
8	·630	·145	1·2604	·1825	·0718	·07196	·0002	·57	·0658	·06580	0	1·094
9	·790	·160	1·3563	·2170	·0935	·09357	·0001	·72	·0831	·08328	·0002	1·124
10	·965	·175	1·4802	·2590	·1194	·11932	·0001	·90	·1039	·10281	·0011	1·161
11	1·150	·185	1·6426	·3040	·1498	·15008	·0003	1·09	·1259	·12440	·0015	1·206
12	1·350	·195	1·8615	·3630	·1861	·18715	·0011	1·30	·1501	·14805	·0020	1·264
1	2	3	4	5	6	7	8	9	10	11	12	13

The work done increases rapidly with the distance of the point of contact from the line of centres, and the result of Reuleaux's method is to compare one curve that is at work a considerable distance from the line with another that is nearer to it.

This is clearly shown by the figures of Reuleaux's comparative examples, for in each case the losses are almost exactly proportional to the arcs of action.

For the purpose of comparison, the two teeth should be taken under precisely the same circumstances, and they should commence work and stop work together. They should have the same arc of action rather than the same addendum, for the addendum has very little to do with the gear except by its effect on the maximum arc of action.

When taken under similar circumstances, involute and cycloidal gear teeth are practically equal with regard to the work done by friction, the difference being always slightly in favor of the involute.

THE LIMITING NUMBERS OF GEAR TEETH.

The treatment of the subject of the limiting numbers of gear teeth is usually so difficult that the student is obliged to take the results as he finds them; for it is a great work of time and patience to follow out the process, and prove the results to be either true or false.

The following processes are easily derived from the trigonometrical conditions of the problem, but I will here give the results only.*

Assume the arc of recess to be a times, and the thickness of the tooth to be b times the circular pitch, and the diametral pitch to be unity. Let d be the number of teeth in the driver, and f the number in the follower.

For the CYCLOIDAL SYSTEM, assume the diameter of the describing circle to be q times the diameter of the follower, and the limiting numbers of teeth will be involved in the following equation:—

$$\frac{f\,q}{d} = \cfrac{1}{\cfrac{\sin\left[\dfrac{360°}{d}\left(a - \dfrac{b}{2}\right) + \dfrac{360°}{f}\dfrac{a}{q}\right]}{\sin\dfrac{360°}{d}\left(a - \dfrac{b}{2}\right)} - 1} \tag{1}$$

which is insoluble in general terms, but from which either f or q can easily be separated, for any particular case, by a few numerical trials.

For a common example, assume the driver to have six teeth, the arc of recess to be equal to the pitch, the tooth to be equal to the space, and the flanks of the follower to be radial. This gives $a = 1$, $b = \frac{1}{2}$, $q = \frac{1}{2}$, and $d = 6$; so that the formula becomes

$$f = \frac{12}{\left(\sin\ +\ \cos\right)\dfrac{720°}{f} - 1} \tag{2}$$

To solve this, put f equal to two numbers as near truth as can be estimated, say 140 and 160. This gives $140 = 140.171$, and $160 = 159.193$, the opposite errors showing that f is between the two chosen points.

Interpolating in proportion to the two errors, we get 143.5 as our first approximation.

Trying 143 and 144 in the same way, we get 143.491 as a second approximation, and 144 as the required nearest larger integer.

If the chosen points had been 120 and 180, the first approximation would have been 144.5, and a second trial would have fixed 144 as the nearest integer.

When the driver is a rack, we must use the formula

$$f = \frac{2\pi\left(a - \dfrac{b}{2}\right)}{q\sin\dfrac{360°}{f}\dfrac{a}{q}} \tag{3}$$

and when the rack is driven we must use

$$d = \frac{2\,a\,\pi}{\tan\dfrac{360°}{d}\left(a - \dfrac{b}{2}\right)} \tag{4}$$

which are simpler than the unlimited formula.

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*This subject I have treated in full, with illustrations and examples, in a paper in the Scientific American Supplement, Vol. XXIII., 1887.

When the INVOLUTE SYSTEM is to be treated, the problem is a double one; for the action on one side of the line of centers will set one limit, while that on the other side will set another.

If we know Q, the angle of obliquity, we have

$$f = 2\,a\,\pi\,\cot Q \tag{5}$$

so that the problem is reduced to finding the value of Q for the given conditions.

The solution is exact, and not dependent, as with cycloidal teeth, on a process of trial and error.

On the approach side we have the formula

$$\tan Q = \frac{2\,\pi}{d}\left(1 - a\right) \tag{6}$$

and on the recess side the formula

$$\cos Q = \sqrt{\frac{p\,\cot\,W + \frac{1}{2} + \sqrt{p\,\cot\,W - p^2 + \frac{1}{4}}}{1 + \cot^2\,W}} \tag{7}$$

in which $p = \dfrac{d}{2\,a\,\pi}$ (8)   and $W = \dfrac{360°}{d}\left(a - \dfrac{b}{2}\right)$ (9)

The approach will set a minimum value for Q, and the recess will determine a maximum. The maximum must evidently be no less than the minimum.

When the involute rack follows, we have the same case as for a cycloidal pinion and rack, see (4); but when the rack drives we can use

$$\cos Q = \sqrt{1 - \frac{b}{2\,a}} \tag{10}$$

The direct solution is somewhat tedious in application, and may be simplified by the use, on the recess side, of the formula

$$\cos Q = \frac{p\,\sin\,W}{\cos\,(Q + W)} \tag{11}$$

which can be easily worked by the above-described process of trial and error.

This supposes the involute to be for the interchangeable system, but when it can be allowed to be non-interchangeable the angle of obliquity on the approach need not be the same as that on the recess. The interchangeable involute tooth will not permit as small pinions as the non-interchangeable cycloidal tooth, but when both forms are taken on the same terms, both non-interchangeable, the advantage of the cycloidal tooth is destroyed.

CONIC PITCH LINES.

The utility of the conic sections, used as the pitch lines of gear wheels, lies in the fact that under certain conditions they will roll together in perfect rolling contact when mounted upon fixed centers.

We can put all the conic sections under one law as to each of several features when rolling together, as follows:

Any two equal conic sections will roll together in perfect rolling contact when fixed on centers at their opposite foci.

Their moving foci will move at a fixed distance apart.

The two curves will make a continuous and complete revolution on each other.

The point of contact of the the two curves will be at the intersection of the line of the fixed foci with the line of the moving foci.

The common tangent to the two curves at their point of contact will pass through the point of intersection of the two axes.

There are four conic sections, varying principally as to their focal distance. The circle, having an infinitely small focal distance; the ellipse, having a finite and positive focal distance; the parabola, having an infinitely great focal distance; and the hyperbola, having a finite and negative focal distance.

Any two curves that will roll together may be used as the pitch lines of gear wheels, and therefore we can have gears with either circular, elliptic, parabolic, or hyperbolic pitch lines.

In either case the moving foci may be connected by a link that will hold the two gears together when in motion, and this link will act in the most direct and advantageous manner when most needed, when the action of the teeth becomes so oblique as to be of little service.
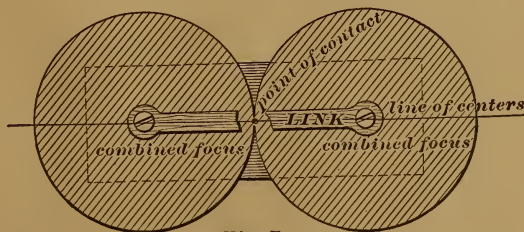
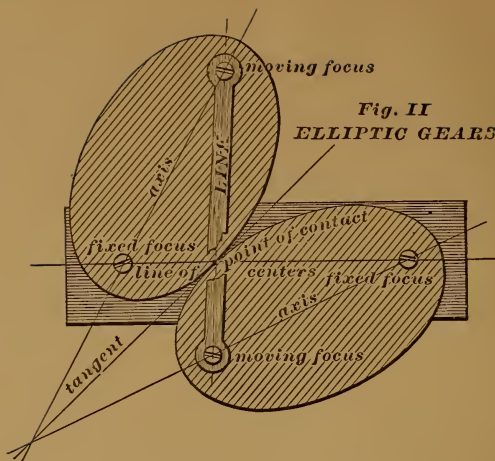The four cases are illustrated by the four figures:
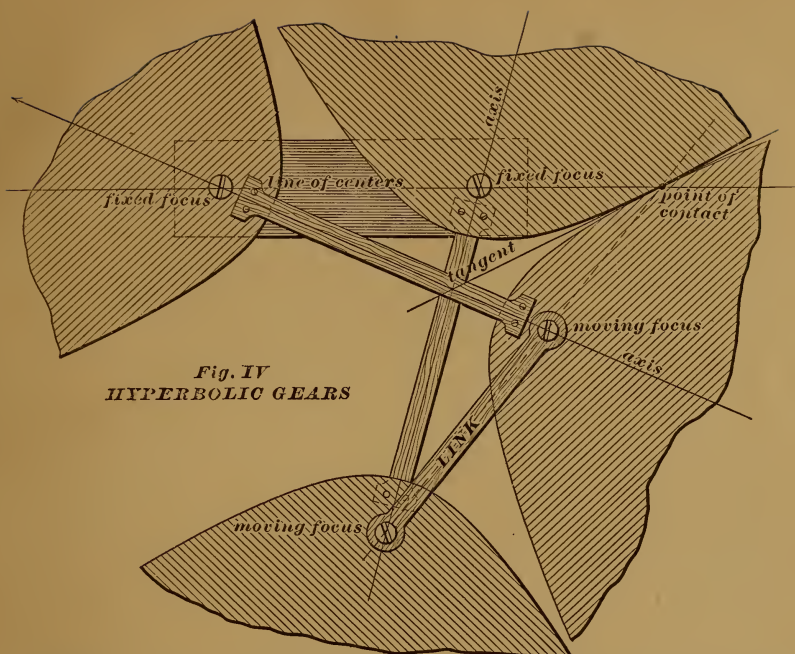

Fig. I
CIRCULAR GEARS

Case I. — When the focal distance is infinitely small the curves are circles, as in figure I. The link is here simply a fixed bar connecting the two centers, for the two foci are combined in one point at the center.

Fig. II
ELLIPTIC GEARS

Case II.   When the focal distance is finite and positive, the curves are ellipses that will roll together if fixed on centers at their opposite foci, as in Fig. II.   The link is a moving bar connecting the two moving foci.

Fig. III
PARABOLIC GEARS
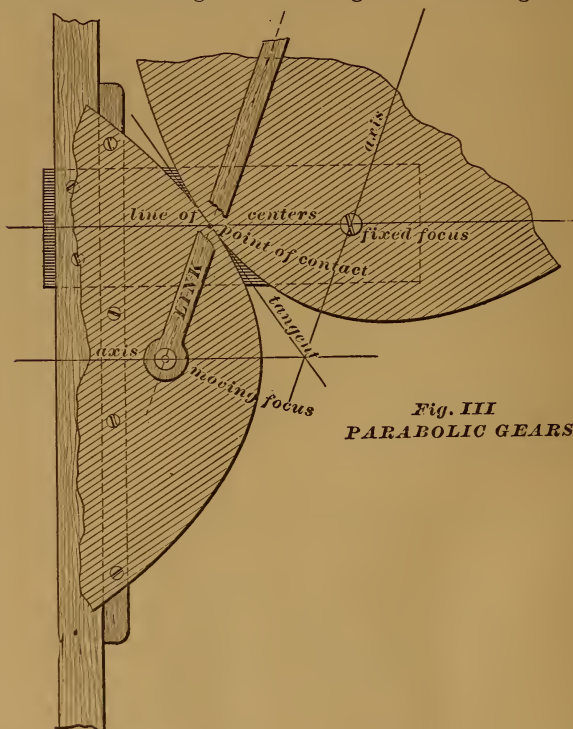
Case III.   When the focal distance is infinitely great the curves are parabolas.   One parabola turns about its focus while the other turns about its opposite focus, but, as the opposite focus is at an infinite distance the second parabola must move in a straight line at right angles to the line of centers, as in Fig III.   The link becomes a bar of infinite length, and cannot be practically applied.   The revolution is complete but of infinite extent, so that it cannot be practically accomplished.

Fig. IV
HYPERBOLIC GEARS

CASE IV. When the focal distance is finite and negative, the curves are hyperbolas. The opposite focus about which one hyperbola turns is now on the other side of the curve, which becomes a negative or internal pitch line, as in Fig IV. The link is of finite length and can be practically applied. The revolution is complete, for as soon as one pair of curves separate, the other pair come together, and the motion is continued.

The utility of circular gears is universal, and elliptic gears have many applications, but no use is apparent for parabolic or hyperbolic gears. A use for them will probably be found when their existence and properties become well known, and they ar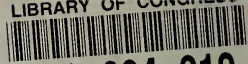e certainly of interest to the student of mechanism.